I-2 FINITE REDUCTIVE GROUPS

From now on, all algebraic varieties are defined over $k = \overline{\mathbb{F}_p}$

1) First example

How to construct the finite group $GL_n(q) := GL_n(\mathbb{F}_q)$
from the algebraic gp $GL_n(\overline{\mathbb{F}_p})$?

Consider $F : GL_n(\overline{\mathbb{F}_p}) \longrightarrow GL_n(\overline{\mathbb{F}_p})$

$$(a_{ij}) \longmapsto (a_{ij}^q)$$

It is an endomorphism of the algebraic group $GL_n(\overline{\mathbb{F}_p})$
called a Frobenius endomorphism

then $$\boxed{GL_n(q) = GL_n(\overline{\mathbb{F}_p})^F}$$

Recall that the coordinate ring of GL_n is $\overline{\mathbb{F}_p}[x_{ij}, \det^{-1}]$
The Frobenius endomorphism is defined by $x_{ij} \longmapsto x_{ij}^q$
which we can write as

$$\overline{\mathbb{F}_p} \otimes_{\mathbb{F}_q} \mathbb{F}_q[x_{ij}, \det^{-1}] \longrightarrow \overline{\mathbb{F}_p} \otimes_{\mathbb{F}_q} \mathbb{F}_q[x_{ij}, \det^{-1}]$$

$$\lambda \otimes P \longmapsto \lambda \otimes P^q$$

the q-th power on some $\mathbb{F}_q$-form of the coordinate ring

2) General case : $\mathbb{F}_q$-structures

def: An affine variety $X = \mathrm{Spec} A$ is defined over $\mathbb{F}_q$ if there is an $\mathbb{F}_q$-subalgebra A_0 of A s.t

$$A = \overline{\mathbb{F}}_p \otimes_{\mathbb{F}_q} A_0$$

The Frobenius endomorphism $F : X \to X$ attached to this structure is defined by

$$\overline{\mathbb{F}}_p \otimes_{\mathbb{F}_q} A_0 \longrightarrow \overline{\mathbb{F}}_p \otimes_{\mathbb{F}_q} A_0$$
$$\lambda \otimes a \longmapsto \lambda \otimes a^q$$

If $G \subseteq GL_n$ is a closed subgroup of GL_n such that $F(G) \subseteq G$ for $F : (a_{ij}) \mapsto (a_{ij}^q)$ [the "standard" Frobenius]

then $F_{|G}$ is a Frobenius endomorphism of G.

Prop: Let F be a Frobenius on X w.r.t some $\mathbb{F}_q$-structure.

(i) F^n is a Frobenius for some $\mathbb{F}_{q^n}$-structure

(ii) X^F is finite

(iii) If $\varphi \in \mathrm{Aut}(X)$ is such that $(\varphi F)^n = F^n$ for some $n \geq 1$ then φF is a Frobenius w.r.t some $\mathbb{F}_q$-structure

(iv) If F' is another Frobenius w.r.t some $\mathbb{F}_q$-structure then $\exists n \geq 1$ s.t. $F^n = F'^n$.

Exercise: determine all the Frobenius endomorphisms of $\mathbb{A}_1$

Ex: $F : GL_n(\overline{\mathbb{F}}_p) \longrightarrow GL_n(\overline{\mathbb{F}}_p)$
$$(a_{ij}) \longmapsto (a_{ij}^q)$$
and $\varphi : \quad M \longmapsto {}^tM^{-1}$ involution of $GL_n(\overline{\mathbb{F}}_p)$
Then $F' = \varphi \circ F = F \circ \varphi$ is also a Frobenius since $F'^2 = F^2$

$$GL_n(\overline{\mathbb{F}}_p)^{F'} = \{ M \in GL_n(\overline{\mathbb{F}}_p) \mid M^{-1} = {}^tF(M) \} \subseteq GL_n(q^2)$$
$=:$ $GU_n(q)$ general finite unitary group

Ex: Let T be a r-dimensional torus and $F : t \longrightarrow t^q$ be the "standard" Frobenius endomorphism

With $T \simeq (\mathbb{G}_m)^r$ we have $F : (\mathbb{G}_m)^r \longrightarrow (\mathbb{G}_m)^r$
$$(t_1, \dots, t_r) \longmapsto (t_1^q, \dots, t_r^q)$$
so that $T^F \simeq (\mathbb{F}_q^\times)^r$ (we say that T is split)

Let $\varphi : (t_1, \dots, t_r) \mapsto (t_2, \dots, t_r, t_1)$ and $F' = F \circ \varphi = \varphi \circ F$

Then $T \twoheadrightarrow \mathbb{G}_m$ induces $T^{F'} \simeq \mathbb{F}_{q^n}^\times$
$(t_1, \dots, t_r) \mapsto t_1$

3) The Lang-Steinberg theorem

Thm: let G be a connected algebraic group. Then the Lang map $G \longrightarrow G$, $g \longmapsto g^{-1}F(g)$ is surjective

This fundamental result has many applications

Corollary: let G be a connected algebraic gp acting on X and F a Frobenius of G and X such that $F(g\cdot x) = F(g)\cdot F(x)$ $\forall g \in G, x \in X$. Then every F-stable orbit of X under G has at least one F-stable point

proof: Assume that $F(x) = g\cdot x$
Write $g^{-1} = h^{-1}F(h)$ for some $h \in G$
Then $F(h\cdot x) = F(h)g\cdot x = h\cdot x$ $\square$

4) Finite reductive groups

def: a finite reductive group is a finite group G^F where
- G is connected reductive alg. group / $\overline{\mathbb{F}_p}$
- $F : G \longrightarrow G$ is a Frobenius endomorphism

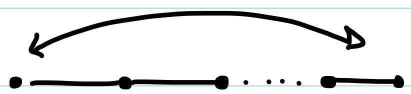
Recall that pairs $T \subseteq B$ where T is a max torus of G
B is a Borel subgp of G
form a single conjugacy class under G.
$\rightsquigarrow$ $\exists$ such a pair with $F(T) = T$ and $F(B) = B$
in that case T is said to be quasi-split

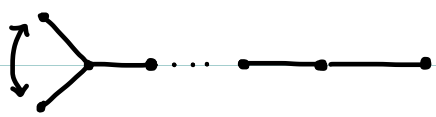
Ex: $T = \begin{pmatrix} * & & \cdot \\ & \ddots & \\ \cdot & & * \end{pmatrix} \subseteq B = \begin{pmatrix} * & & * \\ & \ddots & \\ \cdot & & * \end{pmatrix}$ are stable

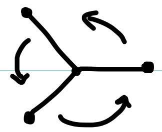
under the "standard" Frobenius $F : (a_{ij}) \longmapsto (a_{ij}^q)$
but not under $F' : M \mapsto {}^t F(M)^{-1}$

Let $T \subseteq B$ F-stable and $W = N_G(T)/T$
Then the action of F on G induces an action on W

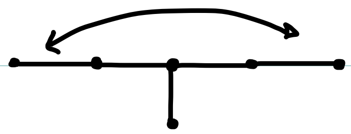
In addition, since $F(B) = B$, the set of simple reflections
$S = \{ w \in W \mid B \cup BwB \text{ is a group} \}$
is permuted by F
$\rightsquigarrow$ automorphism of the Coxeter diagram:


${}^2A_n$ $\quad GU_n(q)$


${}^2D_n$ $\quad SO_{2n}^-(q)$



${}^3D_4$



${}^2E_6$

<u>Rmk</u>: there are groups associated to ${}^2B_2$, ${}^2F_4$, ${}^2G_2$ but only when $p = 2, 2$ or 3 respectively and F is only a root of a Frobenius endomorphism

$\rightsquigarrow$ Ree and Suzuki groups