I-3 THE HECKE ALGEBRA

<u>Motivation</u>: G reductive gp with $F : G \to G$ Frobenius
$B \subseteq G$ F-stable Borel subgroup

Understand the induced representation $\mathrm{Ind}_{B^F}^{G^F} 1_{B^F} = \mathbb{C}G^F/B^F$

1) Bruhat cells and Bruhat decomposition

We start with $G = SL_2 \supseteq B = \left\{ \begin{pmatrix} \lambda & * \\ & \lambda^{-1} \end{pmatrix} \right\} \supseteq T = \left\{ \begin{pmatrix} \lambda & \cdot \\ \cdot & \lambda^{-1} \end{pmatrix} \right\}$

The map $\begin{aligned} G &\twoheadrightarrow \mathbb{P}_1 \\ g &\mapsto g(\mathbb{C}e_1) \end{aligned}$ induces an iso $\begin{aligned} G/B &\xrightarrow{\sim} \mathbb{P}_1 \\ \begin{pmatrix} a & * \\ b & * \end{pmatrix} &\longmapsto [a:b] \end{aligned}$

$\rightsquigarrow \; G^F/B^F \simeq (G/B)^F \simeq \mathbb{P}_1(\mathbb{F}_q)$

The decomposition $\mathbb{P}_1 = \{[1:0]\} \sqcup \{[x:1] \mid x \in \mathbb{A}_1\}$
induces

$$G/B = B/B \sqcup \left\{ \begin{pmatrix} x & * \\ 1 & * \end{pmatrix} B \right\}$$

but $\begin{pmatrix} 1 & x \\ \cdot & 1 \end{pmatrix} \underbrace{\begin{pmatrix} \cdot & 1 \\ -1 & \cdot \end{pmatrix}}_{s} = \begin{pmatrix} -x & \cdot \\ -1 & 1 \end{pmatrix} \implies \begin{pmatrix} x & * \\ 1 & * \end{pmatrix} B = \begin{pmatrix} 1 & x \\ \cdot & 1 \end{pmatrix} s B$

$\implies G = B \sqcup BsB$

In addition $BsBsB \supseteq \begin{cases} B \\ s = \begin{pmatrix} 1 & 1 \\ \cdot & 1 \end{pmatrix} \begin{pmatrix} 1 & \cdot \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ \cdot & 1 \end{pmatrix} \end{cases}$

therefore $BsBsB = B \sqcup BsB \; (= G)$

For the general case, take $T \subseteq B$ $\quad$ T max torus of G
$\quad$ B Borel subgp of G

$\rightsquigarrow W = N_G(T)/_T$, $S \subseteq W$ simple reflections

<u>Thm</u>: (i) $G = \bigsqcup_{w \in W} BwB \leftarrow$ Bruhat cell

Schubert cell

(ii) $BwB/_B \simeq \mathbb{A}_{\ell(w)}$

(iii) $\forall s \in S, w \in W \quad BwBsB = \begin{cases} BwsB & \text{if } \ell(ws) = \ell(w)+1 \\ BwB \sqcup BwsB & \text{otw} \end{cases}$

If in addition $T \subseteq B$ are F-stable then $G^F = \bigsqcup_{w \in W^F} B^F w B^F$

<u>Rmk</u>: let $U = R_u(B)$ so that $B = T \ltimes U$
and $w_0 =$ elt of maximal length in W
then the map $\quad U \cap {}^{ww_0}U \times B \longrightarrow BwB$
$\quad (u, b) \longmapsto uwB$
is an isomorphism (and we recover (ii) since $U \cap {}^{ww_0}U \simeq \mathbb{A}_{\ell(w)}$)

<u>Ex</u>: $G = GL_n$, $U = \begin{pmatrix} 1 & & * \\ & \ddots & \\ \cdot & & 1 \end{pmatrix}$, ${}^{w_0}U = \begin{pmatrix} 1 & & \cdot \\ & \ddots & \\ * & & 1 \end{pmatrix}$

and with $w = (1, 2, \dots, n)$ $\quad U \dot{n}^{ww_0} U = \begin{bmatrix} 1 & * & \cdots & * \\ & 1 & 0 & \cdots & 0 \\ & & 1 & \ddots & \vdots \\ & (0) & & \ddots & 0 \\ & & & & 1 \end{bmatrix} \simeq \mathbb{A}_{n-1}$

2) Order of G^F

We assume for simplicity that F acts trivially on W
$\rightsquigarrow$ we say that (G, F) is split

$\Rightarrow \forall w \in W, \quad \#(BwB/B)^F = q^{\ell(w)}$

We deduce a formula for the order of G^F

$$|G^F| = |B^F| \cdot \# G^F/B^F = |B^F| \left(\sum_{w \in W} q^{\ell(w)} \right)$$

Since $B = T \ltimes U$, $|B^F| = |T^F||U^F| = |T^F| \, q^{\ell(w_0)}$
(note that $\ell(w_0) = \#$ reflections in W)

In addition, if G is semisimple $|T^F| = (q-1)^{\operatorname{rk} T}$

Thm: Assume (G, F) is split and semisimple
Let $(d_1, \dots, d_r)$ be the degrees of W
Then
$$|G^F| = q^{\ell(w_0)} \prod_{i=1}^{r} (q^{d_i} - 1)$$

Ex: • $G = Sl_n$ $(d_1, \dots, d_r) = (2, 3, \dots, n)$ $\ell(w_0) = \frac{n(n-1)}{2}$

$\Rightarrow |Sl_n(q)| = q^{n(n-1)/2} \prod_{i=2}^{n} (q^i - 1)$

• $G = Sp_4$ $(d_1, d_2) = (2, 4)$ $\ell(w_0) = 4$

$\Rightarrow |Sp_4(q)| = q^4 (q^4 - 1)(q^2 - 1)$ $(= |SO_5(q)|)$

3) The induced representation

We give now the structure of $\mathrm{End}_{G^F}(\mathbb{C}G^F/B^F)$

* For representation lovers

Let $e = \frac{1}{|B^F|} \sum_{b \in B^F} b$ be the projection onto the invariant part under the action of B^F

$e^2 = e$ and $eb = be = e$ for all $b \in B^F$

Then (i) $\mathbb{C}G^F/B^F \simeq \mathbb{C}G^F e$

(ii) $\mathrm{End}_{G^F}(\mathbb{C}G^F) \simeq e\,\mathbb{C}G^F e$

(iii) $\{ \underbrace{ewe}_{q^{-\ell(w)} h_w} \}_{w \in W^F}$ basis of $\mathrm{End}_{G^F}(\mathbb{C}G^F)$

* For function lovers

Let X be any finite set with an action of G^F
Then the map

$$\begin{aligned} \mathbb{C}[X\times X] &\longrightarrow \mathrm{End}_{\mathbb{C}}(\mathbb{C}[X]) \\ f &\longmapsto \Big(x \mapsto \sum_{y\in X} f(x,y)y\Big) \end{aligned}$$

induces an isomorphism of algebras

$$\mathbb{C}[X\times X]^{G^F} \xrightarrow{\sim} \mathrm{End}_{G^F}(\mathbb{C}[X])$$

where
- G^F acts diagonally on $X\times X$
- The product in $\mathbb{C}[X\times X]$ is the convolution:

$$(f*f')(x,z) = \sum_{y\in X} f(x,y)f'(y,z)$$

with unit the characteristic function of the diagonal ΔX

Characteristic functions of G^F-orbits on $X\times X$
give a basis of $\mathrm{End}_{G^F}(\mathbb{C}[X])$

Consequence: $h_w :=$ char. function of $G^F\cdot(B^F, wB^F)$
for $X = G^F/B^F$ yield a basis

Prop: Assume F acts trivially on W.
Then the basis $\{h_w\}$ satisfies

$$h_w h_{w'} = h_{ww'} \text{ if } \ell(ww') = \ell(w) + \ell(w')$$

$$h_s^2 = (q-1) h_s + q\, h_1 \quad \text{for } s \in S$$

This is a Hecke algebra with parameter q and group W
We denote it by $\mathcal{H}_q(W)$