I-4 Principal Series Representations

1) Hom-functor

Let V be a representation of G^F (over $\mathbb{C}$)
How much of V is encoded in $\mathrm{End}_{G^F}(V)$?

Thm: The functor
$$\begin{array}{rcl} \mathbb{C}G^F\text{-mod} & \xrightarrow{\Theta} & \mathrm{End}_{G^F}(V)^{op}\text{-mod} \\ M & \longmapsto & \mathrm{Hom}_{G^F}(V, M) \end{array}$$
induces an equivalence between:
- the abelian (semisimple) subcategory of $\mathbb{C}G^F$-mod generated by direct summands of V
- $\mathrm{End}_{G^F}(V)^{op}$-mod

$\rightsquigarrow$ bijection
$$\begin{array}{ccc} \mathrm{Irr}\, V & \xleftrightarrow{1:1} & \mathrm{Irr}\,\mathrm{End}_{G^F}(V) \\ V_E & \longleftarrow & E \end{array}$$
simple direct summand of V $\longrightarrow$ V_E

If we write $V = \bigoplus_E V_E^{n_E}$ then
$$\begin{aligned} n_E &= \dim \mathrm{Hom}_{G^F}(V, V_E) \\ &= \dim \mathrm{Hom}_{\mathrm{End}(V)}(\mathrm{End}(V), E) \\ &= \dim E \end{aligned}$$

$$\rightsquigarrow V = \bigoplus_{E \in \mathrm{Irr}\,\mathrm{End}(V)} V_E^{\oplus \dim E}$$

Corollary : Irreducible summands of $\mathrm{Ind}_{B^F}^{G^F}(1_{B^F})$
$\xleftrightarrow{1:1}$ Irreducible representations of $\mathcal{H}_q(W)$

2) Representations of the Hecke algebra

Generic Hecke algebra

$$\mathcal{H}_x(W) = \left\langle h_w \;\middle|\; \begin{array}{l} h_w h_{w'} = h_{ww'} \text{ if } \ell(ww') = \ell(w)+\ell(w') \\ h_s^2 = (x-1)h_s + x h_1 \end{array} \right\rangle$$

algebra over $\mathbb{C}(\sqrt{x})$

$$\begin{tikzcd} & \mathcal{H}_q(W) \arrow[d, dashed, "\text{“}q=1\text{”}"] & \text{semisimple} \\ \mathcal{H}_x(W) \arrow[ur, "x=q"] \arrow[r, "x=1"'] & \mathbb{C}W & \text{semisimple} \end{tikzcd}$$

actually "$\sqrt{q}=1$"

Thm : The specialization "$q=1$" induces a bijection

$$\begin{array}{ccc} \mathrm{Irr}\,\mathcal{H}_q(W) & \xleftrightarrow{1:1} & \mathrm{Irr}_{\mathbb{C}} W \\ \chi_q & \longleftarrow & \chi \end{array}$$

s.t. $\chi_q(h_w)_{q=1} = \chi(w)$.

Ex : $T_w \longmapsto q^{\ell(w)}$ is the q-deformation of 1_W
$T_w \longmapsto (-1)^{\ell(w)}$ ______________ sgn_W

Rmk : the bijection preserves the dimension of irreducible representations.

3) Unipotent principal series (recall that we assume $W^F = W$)

def: an irreducible character of G^F is in the unipotent principal series if it is a constituent of $\mathrm{Ind}_{B^F}^{G^F} 1_{B^F}$

(Note that it does not depend on the choice of B^F)

By the previous results there is a bijection

$$\begin{array}{ccc} \mathrm{Irr}(\mathrm{Ind}_{B^F}^{G^F} 1_{B^F}) & \overset{1:1}{\longleftrightarrow} & \mathrm{Irr}\, W \\ \rho_\chi & \longleftarrow & \chi \end{array}$$

Every unipotent principal series character is of the form ρ_χ for some $\chi \in \mathrm{Irr}\, W$ and

$$\mathrm{Ind}_{B^F}^{G^F}(1_{B^F}) = \sum_{\chi \in \mathrm{Irr} W} \chi(1)\, \rho_\chi$$

4) Examples

a) $G = SL_2$ $W = \mathbb{Z}/2\mathbb{Z}$ has two irreducible representations namely 1_W and sgn_W

$$\mathbb{C}G^F/B^F = \mathbb{C}\mathbb{P}_1(\mathbb{F}_q) = \rho_1 + \rho_{\mathrm{sgn}}$$

since $\sum_{x \in \mathbb{P}_1(\mathbb{F}_q)} x$ is invariant under $SL_2(q)$ one of the irrep is trivial, the other has dim $= q+1-1 = q$

Actually $\rho_1 = 1_{G^F}$ and $\dim \rho_{sgn} = q$

More generally, for a general finite reductive group we have $\rho_1 = 1_{G^F}$ and $\dim \rho_{sgn} = q^{\ell(w_0)}$

called the Steinberg character St_{G^F}

b) $G = GL_3$, $W = \mathfrak{S}_3$ has 3 conjugacy classes hence 3 irreducible representations namely 1_W, sgn_W and χ with $1^2 + 1^2 + \chi(1)^2 = |\mathfrak{S}_3| = 6$
$\Rightarrow$ χ has dimension 2

$$\mathrm{Ind}_{B^F}^{G^F} = \underbrace{1_{G^F}}_{\dim 1} + \underbrace{St_{G^F}}_{\dim q^3} + 2\rho_\chi \qquad \dim 1 + 2q + 2q^2 + q^3$$

$\Rightarrow$ $\dim \rho_\chi = q(q+1)$

<u>Rmk</u>: one can compute $\dim \rho_\chi$ inside $\mathcal{H}_q(W)$ using Schur elements. Then $\dim \rho_\chi$ is a polynomial in q dividing $|G^F|$ as a polynomial.

c) $G = GL_n$ $\quad W = \mathfrak{S}_n$

$\mathrm{Irr}\, W \longleftrightarrow$ partitions of n

$\chi_\lambda \longleftarrow\!\shortmid\ \lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \dots \geqslant 0)$ with $\sum \lambda_i = n$

with $\chi_{(n)} = 1_{\mathfrak{S}_n}$ and $\chi_{(1^n)} = \mathrm{sgn}_{\mathfrak{S}_n}$

$\Rightarrow$ unipotent principal series representations of $GL_n(q)$ are parametrized by partitions $\{\rho_\lambda\}_{\lambda \text{ part. of } n}$

<u>Ex</u>: $\rho_{(n)} = 1_{GL_n(q)}$, $\rho_{(1^n)} = \mathrm{St}_{GL_n(q)}$ of dim $q^{n(n-1)/2}$

<u>Rmk</u>: there is an explicit formula for $\dim \rho_\lambda$ which is a q-analogue of the hook length formula for $\dim \chi_\lambda$