§1 Flag Vtys in type A (GL_n)

1.1 Defns E a v.s$/k$, $\dim E = n$.

Def: A (complete) flag in E is a sequence of subspaces
$$V_\bullet = (0 = V_0 \subset V_1 \subset V_2 \subset \dots \subset V_n = E) \quad \text{s.t. } \dim V_i = i$$

The set $\mathcal{B}$ of all flags is the flag variety.

Ex: $n=2$: $V_\bullet = (0 \subset L \subset E)$ so $\mathcal{B} = \mathbb{P}^1_k = \mathbb{P}(E)$. It's a top. space and an alg vty too.

Ex: $n=3$. Real life flags. [Here we are, in $\mathbb{R}^3$. Plant a flag to mark your territory. The (∞-ly long) flagpole gives a line thru the origin (center of the earth). The flag gives a plane containing the flagpole!

The (flagpole $\subseteq$ flag) variety.

As flag flutters in breeze or as flag moves around, you move around in the top. space $\mathcal{B}$.

Another visualization: A flicker/spinner on a game board.

§1.2 Action Def: An ordered basis (or obasis) $(e_1, \dots, e_n)$ for E is adapted to $V_\bullet$

if $V_1 = \operatorname{Span} e_1 = \langle e_1 \rangle$; $V_2 = \langle e_1, e_2 \rangle$ ...

Each ordered basis gives one flag.

Each flag has many adapted bases. If $(e_1, e_2, \dots)$ is adapted

so is $(e_1, e_1 + e_2, \dots)$ since $\langle e_1, e_2 \rangle = \langle e_1, e_1 + e_2 \rangle$.

Now $G = GL(E)$ acts simply transitively on ordered bases $\Leftrightarrow$ invertible matrices

acts compatibly on $\mathcal{B}$ via $(gV)_i = g(V_i)$

So $GL(E) \circlearrowright \mathcal{B}$ transitively. ⌈What is stabilizer of a point?⌋

Fix std basis $(x_1, \dots, x_n) \rightsquigarrow$ standard flag Std.

$$\mathrm{Stab}_G(\mathrm{Std}) = \begin{pmatrix} * & * & & * \\ 0 & * & & \\ 0 & 0 & \ddots & \\ 0 & 0 & & \\ 0 & 0 & 0 & * \end{pmatrix} = B \quad \text{the standard Borel.}$$

b/c $x_1 \mapsto \in \langle x_1 \rangle$

b/c $x_2 \mapsto \in \langle x_1, x_2 \rangle$

Hence $\mathcal{B} \cong G/B$. (Rmk: This makes $\mathcal{B}$ an alg vty. Other ways too — Plücker embedding — less useful to us.)

$g \cdot \mathrm{Std} \leftarrow gB$

By Lie-Kolchin we have

$$GL(E) \circlearrowright \mathcal{B}(E) \xleftrightarrow{\sim} \text{Borel subgps of } GL(E) \circlearrowleft_{GL(E)} \text{ by conjugation}$$

$$V_\bullet \longmapsto \mathrm{Stab}(V_\bullet)$$

⌈as expected for stabilizers of points in an orbit.⌋

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§1.3 Comparing flags

Pick two random flags. Don't expect same flagpole, expect generic behavior.

3D: take flag V. Blow on flag, then "flick" pole (new line in same plane) to get V'. Are they generic? No: line of V' is still in plane of V, but generically $\dim(V_1' \cap V_2) = 1+2-3 = 0$

$\rightsquigarrow$ use $\dim V_i \cap V_j'$ to compare.

Ex: Let $w \in S_n \subset GL(n)$. (via std basis) What is $\mathrm{Std}_i \cap (w \cdot \mathrm{Std})_j$?

$$= \langle x_1, \dots, x_i \rangle \cap \langle x_{w(1)}, \dots, x_{w(j)} \rangle = \langle x_k \mid k \in \{1 \dots i\} \cap \{w(1) \dots w(j)\} \rangle$$

so $\dim \mathrm{Std}_i \cap (w \cdot \mathrm{Std})_j = \# \{1 \dots i\} \cap \{w(1) \dots w(j)\} =: d^w_{ij}$.

Prop: Let $V, V' \in \mathcal{B}$. Then $\exists!\, w \in S_n$ s.t. $\exists$ obasis $(e_1, \dots, e_n)$ for V with $(e_{w(1)}, \dots, e_{w(n)})$ adapted to V'. Equivalently, $\exists!\, w \in S_n$ s.t. $\dim (V_i \cap V'_j) = d^w_{ij}$ $\forall i, j$. We say (V, V') are in relative position w and write $V \xrightarrow{w} V'$.

Compatibility w/ action: If $g \in GL(E)$ and $V \xrightarrow{w} V'$ then $gV \xrightarrow{w} gV'$

since $\dim V_i \cap V'_j = \dim g(V_i \cap V'_j) = \dim (gV)_i \cap (gV')_j$.

Conversely if $V \xrightarrow{w} V'$ then $\exists g$ sending $(e_1 \dots e_n)$ to $(x_1 \dots x_n)$

so $(gV, gV') = (\mathrm{Std}, w \cdot \mathrm{Std})$

as in prop ($\Rightarrow (e_{w(1)} \dots e_{w(n)})$ to $(x_{w(1)} \dots x_{w(n)})$)

Consequence: The map $\mathcal{B} \times \mathcal{B} \to S_n$

$$(V, V') \mapsto w \quad \text{s.t.} \quad V \xrightarrow{w} V'$$

induces a bijection b/w $GL(E)$ orbits on $\mathcal{B} \times \mathcal{B}$ and S_n.

$$\mathcal{O}(w) \longleftarrow w$$

Ex: $n=2$. $\mathbb{P}^1 \times \mathbb{P}^1 \ni (L, L')$ has 2 orbits:

$w=1$ $\quad L = L'$ $\quad L \xrightarrow{1} L'$	$w = s = (1\,2)$ $\quad L \neq L'$ $\quad L \xrightarrow{s} L'$.
so $\mathcal{O}(1) = \Delta\mathbb{P}^1 \subset \mathbb{P}^1 \times \mathbb{P}^1$	$\mathcal{O}(s)$ = the rest

Rmk: $L \xrightarrow{s} L' \xrightarrow{s} L''$ what about $L \xrightarrow{?} L''$?

Could be either!

<u>Ex:</u> $n=3$ on exercises. 6 orbits, ways in which flags can interact.

<u>Big Warning:</u> S_n does NOT act on $\mathcal{B}$!

S_n acts on obases to change order. But for $w \in S_n$,

$$\text{flag} \begin{matrix} \nearrow \text{obasis} \to w\cdot\text{obasis} \to \text{flag} \\ \ne \\ \searrow \text{obasis} \to w\cdot\text{obasis} \to \text{flag} \end{matrix} \quad \text{possible.}$$

$w \cdot \mathrm{Std}$ makes sense b/c we made a choice of obasis, but not $w \cdot V$.

$V \xrightarrow{w} V'$ does NOT mean "$V' = w(V)$"!!

⌜ $L \xrightarrow{s} L'$, L' can be lots of things, not just "$s(L)$"... ⌟

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§2 General Case

G a connected reductive group.

§2.1 Defns, etc.

<u>Def:</u> The <u>flag variety of G</u> is the set $\mathcal{B}$ of all Borel subgroups.

<u>Rmk:</u> $G \circlearrowright \mathcal{B}$ by conjugation. $\mathrm{Stab}_G(B) = N_G(B) = B$ so $\mathcal{B} \cong G/B$
$gBg^{-1} \leftarrow\!\shortmid\, gB$

Like fixing a standard basis, let us fix $T \subset B$, and recall $W = N(T)/T$.

<u>Prop:</u> The following are in bijection.

$$W \longleftrightarrow G \text{ orbits on } \mathcal{B}\times\mathcal{B} \longleftrightarrow B \text{ orbits on } G/B \longleftrightarrow B\times B \text{ orbits on } G$$

$$w \longmapsto \begin{matrix}\mathcal{O}(w) = \\ \text{orbit of}\end{matrix} (B, wBw^{-1}) \longleftrightarrow BwB/B \longleftrightarrow BwB$$

content of prop. | abstract nonsense.

<u>Rmk:</u> Left we know, this is Bruhat decomposition.

Note: Again W does NOT act on $\mathcal{B}$. But $\underset{\in G/B}{wB}$ (or $\underset{\in \mathcal{B}}{wBw^{-1}}$) still makes sense. For $w \in N(T)/T$, if $\tilde{w} \in N(T)$, then $\tilde{w}B$ makes sense, indep. of lift since $T \subset B$. [But T is not in every Borel.]

Def: (B_1, B_2) in relative position w if $(B_1, B_2) \in \mathcal{O}(w)$, write $B_1 \xrightarrow{w} B_2$.

§2.2 [Connection to Coxeter theory]

Key Properties: ① $\dim \mathcal{O}(w) = \dim \mathcal{B} + \ell(w) \iff \dim BwB/B = \ell(w)$

$\mathcal{O}(w) \subset \mathcal{B}\times\mathcal{B}$, $\quad BwB/B \subset \mathcal{B}$

② If (*) $\ell(w) + \ell(w') = \ell(ww')$ then

$$B_1 \xrightarrow{w} B_2 \xrightarrow{w'} B_3 \Rightarrow B_1 \xrightarrow{ww'} B_3$$

Rmk: Need (*), $L \xrightarrow{s} L' \xrightarrow{s} L''$ $\not\Rightarrow L \xrightarrow{1} L''$

Moreover

$$\mathcal{O}(w) \times_{\mathcal{B}} \mathcal{O}(w') \xrightarrow{\sim} \mathcal{O}(ww') \text{ is an isomorphism.}$$

$$((B_1, B_2), (B_2, B_3)) \longmapsto (B_1, B_3)$$

same

We'll explore for $GL(3)$ in exercises.

Now, an orbit closure is a union of orbits (general nonsense) so

$$\overline{\mathcal{O}(w)} = \coprod_{\text{some } v} \mathcal{O}(v).$$

Def: $v \le w$ if $\mathcal{O}(v) \subseteq \overline{\mathcal{O}(w)}$

$\iff \overline{\mathcal{O}(v)} \subseteq \overline{\mathcal{O}(w)}$.

Called the Bruhat (partial) order.

Ex: $n=2$. $\mathcal{O}(1) = \Delta\mathbb{P}^1$ is closed

$\overline{\mathcal{O}(s)} = \mathbb{P}^1 \times \mathbb{P}^1 = \mathcal{O}(s) \cup \mathcal{O}(1)$ $\quad 1 \le s$.

Lines can "snap together" in closure.

Exer: Agrees w/ "subexpression definition" of Bruhat order.

Ex: For $GL(E)$: closure can make $\dim(V_i \cap V_j')$ increase.

Exer: Verify that $v \le w \iff d_{ij}^v \ge d_{ij}^w \ \forall i,j$.

Again, we'll explore for $GL(3)$ in exercises.

Rmk: $\leq$ has ! minimal element 1. $\mathcal{O}(1) = \Delta\mathcal{B} \subset \mathcal{B}\times\mathcal{B}$, closed

! maximal element w_0. $\mathcal{O}(w_0) =$ "generic position" $\overline{\mathcal{O}(w_0)} = \mathcal{B}\times\mathcal{B}$.

Rmk: $\mathcal{O}(w)$ is an orbit so it is smooth. (Can't be singular everywhere, but every point in an orbit is the same.)

$\overline{\mathcal{O}(w)}$ is typically singular.

Studying singularity of $\overline{\mathcal{O}(w)}$ at a point on $\mathcal{O}(v)$ is (a part of)

Kazhdan–Lusztig theory. A different WARTHOG – you missed it.