3. Flag Vhs in type A (G_2)

1.1. Define $E$ a vs/k, dim $E = n$.

**Def:** A (complete) **flag in** $E$ is a sequence of subspaces

$$V_0 = \begin{cases} \{0\} & : V_0 \subset V_1 \subset V_2 \subset \ldots \subset V_n = E \end{cases}$$

such that $dim V_i = i$

The set $\mathcal{B}$ of all flags is the flag variety.

**Ex:** $n = 2$ : $V_0 = (0 < L < E)$ so $\mathcal{B} = P_1 = \text{IP}(E)$. It’s a top space and an alg. variety too.

**Ex:** $n = 3$. Real life flags. Here we are, in $\text{IP}^3$. Plant a flag to mark your territory. The (infinitely long) flagpole gives a line thru the origin (center of the earth). The flag gives a plane containing the flagpole!

The (flagpole $\leq$ flag) variety.

As flag flutter’s in breeze or as flag waves around, you move around in the top.

Another visualization: A Frisbee/spinner on a game board.

3.1.2. **Action Def:** An ordered basis $(e_1, e_2, \ldots)$ for $E$ is adapted to $V$.

If $V_1 = \text{Sp} e_1 \equiv (e_1)$, $V_2 = \langle e_1, e_2 \rangle$ ...

Each ordered basis gives one flag.
Each flag has many adapted bases. If $(e_1, e_2, \ldots)$ is adapted

so is $(e_1, e_2, \ldots)$ since $\langle e_1, e_2 \rangle = \langle e_1, e_2, e_3 \rangle$. 


Now \( G = \text{GL}(E) \) acts simply transitively on ordered basis \( \leftrightarrow \) invertible matrices, and compatibly on \( B \) via \( (gV)_i = g(v_i) \).

So \( \text{GL}(E) \) acts transitively on \( B \). «What is stabilizer of a point?»

Fix standard \( (e_1, \ldots, e_n) \rangle \mapsto \text{standard flag} \ Stab.

\[
\text{Stab}_G(Stab) = \begin{pmatrix}
\cdots & x & \cdots \\
\cdots & 0 & \cdots \\
\cdots & 0 & \cdots \\
\end{pmatrix} = B \quad \text{the standard basis}
\]

\[\begin{align*}
&v_1 \mapsto e \langle e_1 \rangle \\
&x_1 \mapsto e \langle e_1, e_2 \rangle \\
&y_1 \mapsto e \langle e_1, e_2, e_3 \rangle
\end{align*}\]

Hence \( B \cong G/B \). «This makes \( B \) an alg. vty. Other ways too.»

\[g \cdot Stab \leftarrow gB\]

Ruler embeds — less useful to us.

By Lie-Kolchin we have

\[\text{GL}(E) \cong \text{Bord subgps of } \text{GL}(E)\]

\[\begin{array}{c}
\text{GL}(E) \cong \\
\text{Bord subgps of } \text{GL}(E)
\end{array}\]

By configuration

\[\text{GL}(E) \text{ by configuration}\]

\[\text{as expected for stabilizers of points in an orbit.}\]

\[\text{3.3 Comparing flags} \]

Pick two random flags. Don't expect same flagspace.

Expect generic behavior.

3D: take flag \( V \). Blow on flag, then "flick" pole (new line in same plane)

to get \( V' \). Are they generic? No: line of \( V' \) is still in plane of \( V \), but generically \( \dim(V'_1 \cap V_2) = 1+2-3 = 0 \)

Hence use \( \dim V_1 \cap V'_2 \) to compare.
Ex: Let $w \in S_n \in GL(n)$. (na stel basis) What is $Stab\ {n} (w)$? ③

$$= \langle x_1, \ldots, x_i \rangle \cap \langle x_{w(1)}, \ldots, x_{w(n)} \rangle$$

So $\dim Stab\ {n} (w) = \# \{ i \mid 1 \leq i \leq n \cap w(i) = i \} = \cdot d_i^w$.

Prop: Let $V, V' \in B$. Then $\exists! w \in S_n$ s.t. $w$ obtains $(e_1, \ldots, e_n)$

for $V$ with $(e_{w(1)}, \ldots, e_{w(n)})$ adapted to $V'$. Equivalently, $\exists! w \in S_n$

s.t. $\dim (V_i \cap V'_j) = d_i^w \forall i, j$. We say $(V, V')$ are in

relative position $w$ and write $V \rightarrow^w V'$.

Compatibility $w$ action: If $g \in GL(E)$ and $V \rightarrow^w V'$ then $gV \rightarrow^w gV'$

since $\dim V_i \cap V'_j = \dim g(V_i \cap V'_j) = \dim (gV_i \cap gV'_j)$.

Conversely if $V \rightarrow^w V'$ then $\exists g$ sending $(e_1, \ldots, e_n)$ to $(e_{g(1)}, \ldots, e_{g(n)})$

so $(gV, gV') = (Stab\ {n} (w) \cdot Std)$.

Consequence: The map $B_k \otimes B \rightarrow S_n$

$(V, V') \mapsto w$ s.t. $V \rightarrow^w V'$

induces a bijection btw $GL(E)$ orbit on $B_k \otimes B$ and $S_n$.

Ex: $n=2$. $P^{1} \times P^{1}$ $\cong (L_0, L_1^*)$ has 2 orbits:

$w=1 \quad L=L' \quad L \rightarrow L'$ $w=\sigma(1,2) \quad L \neq L' \quad L \rightarrow L'$

so $\Delta P^{1} \cdot c P^{1}$ $\cong S_2$ and the rest

Rmk: $L \rightarrow L'$, what about $L \rightarrow L''$? Could be either!
Ex: \( n = 3 \) on Exercise. 6 orbits, ways in which flags can interact.

**Big Warning!** \( \text{S}_n \) does NOT act on \( \mathbb{B} \)!

\( \text{S}_n \) acts on obuses to change order. But for \( \text{w} \in \text{S}_n \),

\[
\begin{align*}
\text{flag} & \rightarrow \text{obast} \rightarrow \text{wo obasi} \rightarrow \text{flag} \\
\text{wo obasi} & \rightarrow \text{v. obasi} \rightarrow \text{flag}
\end{align*}
\]

\( \text{wo Stel} \) makes sense by we made a choice, but not \( \text{wo V} \) of obasii.

\( V \rightarrow V' \) does NOT mean "\( V' = \text{w}(V) \)!!"

\( \text{L} \leq \text{L}' \), \( \text{L}' \) can be lots of things, not just "\( s(L) \)"...

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**3.2 General Case**

\( G \) a connected reductive group

**3.2.1 Definition**

Def.: The flag variety of \( G \) is the set \( \mathcal{G} \) of all Borel subgroups.

Let: \( G \mathcal{C} \mathcal{B} \) by conjugation.

\( \text{Stab}_g(B) = N_G(B) = B \) so \( \mathcal{G} \simeq G/\mathcal{B} \)

Like fixing a standard basis, let us fix \( T \in \mathcal{B} \), and recall \( W = N_G(T)/T \).

**Prop.** The following are in bijection:

\( W \leftrightarrow G \) orbit on \( G \mathcal{C} \mathcal{B} \leftrightarrow B \) orbit on \( G/\mathcal{B} \leftrightarrow B \times B \) orbit on \( G/\mathcal{B} \)

\( w \rightarrow (B, wBu^{-1}) \leftrightarrow BwB/\mathcal{B} \leftrightarrow BwB \)

Ref: Last we know, this is Bruhat decomposition.
Note: Again, W does NOT act on B. But is 85°4

still makes sense. For \(\omega \in \text{V(1)} \) if \(\omega \in \text{V(1)}\), then \(WB\) makes sense

indep of \(W\) since \(\text{TcB}\). [But \(\omega\) is not in every bond.]

\(\text{Def: } (B_1, B_2) \text{ in relative position } W \text{ if } (B_1, B_2) \in \text{O}(w) \text{ where } B_1 \rightarrow B_2.\)

4.2.2 Connection to Cartan Theory

Key Properties:
1. \(\text{dim } \text{O}(w) = \text{dim } \mathbb{B} + l(w) \implies \text{dim } \mathbb{B} \omega = l(w) \)

2. If \(l(w) + l(w') = l(ww')\) then

\[ B_1 \rightarrow B_2 \rightarrow B_3 \rightarrow B_1 \rightarrow B_3 \]

Moreover,

\[ \text{O}(w) \times \text{O}(w') \rightarrow \text{O}(ww') \text{ is an isomorphism.} \]

We'll explore \(\text{O}(\mathbb{B})\) in exercises.

Now, an orbit closure is a union of orbits (general converse) so

\[ \overline{\text{O}(w)} = \bigcup_{\text{same } v} \text{O}(v) \] called the Bruhat (partial) order.

\[ \text{Ex: } n=2, \quad \text{O}(v) = \Delta v^1 \text{ is closed.} \]

\[ \overline{\text{O}(v)} = \{ v' | v' \in \text{O}(v) \} = \text{O}(v) \cup \text{O}(1) \]

\[ \text{Exer: } \text{Agrees w/ "subexpression definition" of Bruhat order.} \]

\[ \text{Exer: } \text{For } \text{GL}(E): \text{ closure can make } \text{dim}(V \cap V') \text{ increase.} \]

\[ \text{Exer: } \text{Verify that } v \leq w \iff d_{ij}^v \geq d_{ij}^w \text{ } \forall ij. \]
Again, we'll explore for $GL(3)$ in exercises.

Rank: $< n$ has 1 minimal element $\lambda$. $O(1) = ABC \otimes_{\lambda} \mathbb{C}$, standard.

Maximal element $\omega_0$. $O(\omega_0) = \text{"generic position"}$ $O(\omega_0) = \mathbb{C} \otimes_{\lambda} \mathbb{C}$

Rank: $O(\omega)$ is an orbit so it is smooth. (Can't be singular everywhere, but every point in an orbit is the same.)

$O(\omega)$ is typically singular.

Studying singularity of $O(\omega)$ at a point on $O(\omega)$ is (a part of) Kazhdan–Lusztig theory.

A different WARTHOG - you nuked it.