

## §1 Flag Vlys in type A ( $\mathbb{C}^n$ )

1.1 Defns |  $E$  a v.s./ $k$ ,  $\dim E = n$ .

Def: A (complete) flag in  $E$  is a sequence of subspaces

$$V_0 = (0 = V_0 \subset V_1 \subset V_2 \subset \dots \subset V_n = E) \text{ s.t. } \dim V_i = i$$

The set  $\mathcal{B}$  of all flags is the flag variety.

Ex:  $n=2$ :  $V_0 = (0 \subset L \subset E)$  so  $\mathcal{B} = \mathbb{P}_k^1 = \mathbb{P}(E)$ . It's a top. space and an alg vty too.

Ex:  $n=3$ . Real life flags. Here we are, in  $\mathbb{R}^3$ . Plant a flag to mark your territory. The (cooly long) flagpole gives a line thro the origin (center of the earth). The flag gives a plane containing the flagpole!

The (flagpole  $\subseteq$  flag) variety.

As flag flutters in breeze or as flag moves around, you move around in the top. space  $\mathcal{B}$ .

Another visualization: A flicker/spinner on a game board.

§1.2 Action | Def: An ordered basis  $(e_1, \dots, e_n)$  for  $E$  is adapted to  $V$ .

$$\text{If } V_1 = \text{Span } e_1 = \langle e_1 \rangle \quad V_2 = \langle e_1, e_2 \rangle \quad \dots$$

Each ordered basis gives one flag.

Each flag has many adapted bases.

If  $(e_1, e_2, \dots)$  is adapted

so is  $(e_1, e_1 + e_2, \dots)$  since  $\langle e_1, e_1 \rangle = \langle e_1, e_1 + e_2 \rangle$ .

Now  $G=GL(E)$  acts simply transitively on ordered basis  $\leftrightarrow$  invertible matrices (2)  
 acts compatibly on  $\mathcal{B}$  via  $(gV)_i = g(V_i)$

So  $GL(E) \curvearrowright \mathcal{B}$  transitively. [What is stabilizer of a point?]

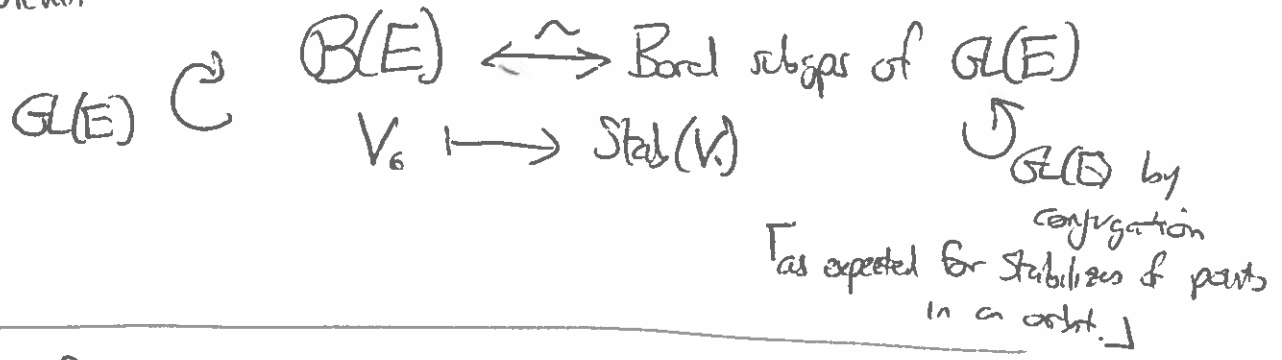
Fix stabobasis  $(x_1, \dots, x_n) \rightsquigarrow$  standard flag  $Std.$

$$Stab_G(Std) = \left( \begin{array}{c|c|c} x_1 & x_2 & \star \\ \hline 0 & x_2 & \star \\ \hline 0 & 0 & \star \\ \hline 0 & 0 & \star \\ \hline 0 & 0 & \star \end{array} \right) = \mathcal{B} \text{ the standard Borel.}$$

b/c  $x_1 \mapsto E \langle x_1 \rangle$  b/c  $x_2 \mapsto E \langle x_1, x_2 \rangle$

Hence  $\mathcal{B} \cong G/B$ .  $g \cdot Std \leftarrow gB$ .  
 (Rank! This makes  $\mathcal{B}$  an alg. vty. Other ways too -  
 Plucker embedding - less useful to us.)

By Lie-Kolchin we have



§1.3 Company flags | Pick two random flags. Don't expect same flagpole, expect generic behavior.

3D: take flag  $V$ . Blow on flag, then "flick" pole (now line in same plane) to get  $V'$ . Are they generic? No: line of  $V'$  is still in plane of  $V$ , but generically  $\dim(V'_1 \cap V_2) = 1+2-3=0$

$\rightsquigarrow$  use  $\dim V_i \cap V'_j$  to compare.

Ex: Let  $w \in S_n \subset GL(n)$  (no std basis) What is  $\text{Std}_E \cap (w \cdot \text{Std}_E)$ ? (3)

$$= \langle x_1, \dots, x_i \rangle \cap \langle x_{w(1)}, \dots, x_{w(j)} \rangle = \langle x_k \mid k \in \{1, \dots, i\} \cap \{w(1), \dots, w(j)\} \rangle$$

so  $\dim \text{Std}_E \cap (w \cdot \text{Std}_E) = \#\{1, \dots, i\} \cap \{w(1), \dots, w(j)\} =: d_{ij}^w$ .

Prop: Let  $V, V' \in \mathcal{B}$ . Then  $\exists!$   $w \in S_n$  s.t.  $\exists$  basis  $(e_1, \dots, e_n)$  for  $V$  with  $(e_{w(1)}, \dots, e_{w(n)})$  adapted to  $V'$ . Equivalently,  $\exists!$   $w \in S_n$  s.t.  $\dim(V_i \cap V'_j) = d_{ij}^w \forall i, j$ . We say  $(V, V')$  are in relative position  $w$  and write  $V \xrightarrow{w} V'$ .

Compatibility w/ action: If  $g \in GL(E)$  and  $V \xrightarrow{w} V'$  then  $gV \xrightarrow{w} gV'$

since  $\dim V_i \cap V'_j = \dim g(V_i \cap V'_j) = \dim (gV_i) \cap (gV'_j)$ .

Conversely if  $V \xrightarrow{w} V'$  then  $\exists g$  sending  $(e_1, \dots, e_n)$  to  $(x_1, \dots, x_n)$   
 so  $(gV, gV') = (\text{Std}_E, w \cdot \text{Std}_E)$  as in prop  $\Rightarrow (e_{w(1)}, \dots, e_{w(n)})$  to  $(x_{w(1)}, \dots, x_{w(n)})$

Consequence: The map  $\mathcal{B} \times \mathcal{B} \rightarrow S_n$   
 $(V, V') \mapsto w$  s.t.  $V \xrightarrow{w} V'$

induces a bijection b/w  $GL(E)$  orbits on  $\mathcal{B} \times \mathcal{B}$  and  $S_n$ .  
 $\mathcal{O}(w) \longleftarrow w$

Ex:  $n=2$ .  $\mathbb{P}^1 \times \mathbb{P}^1 \ni (L_0, L'_0)$  has 2 orbits:

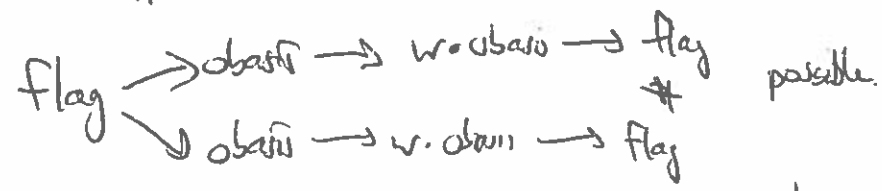
$$w=1 \quad L=L' \quad L \xrightarrow{1} L' \quad \left| \quad w=s=(12) \quad L \neq L' \quad L \xrightarrow{s} L' \right.$$

~~$\mathcal{O}(s)$~~   $\Delta \mathbb{P}^1 \subset \mathbb{P}^1 \times \mathbb{P}^1$   $\mathcal{O}(s)$  is the rest

Rmk!  $L \xrightarrow{s} L' \xrightarrow{s} L''$  what about  $L \xrightarrow{?} L''$ ?  
 Could be either!

Ex:  $n=3$  on exercises. 6 orbits, ways in which flags can interact. (4)

Big Warning:  $S_n$  does NOT act on  $B!$   
 $S_n$  acts on bases to change order. But for w in  $S_n$ ,



w.stad makes sense b/c we made a choice, but not  $w \cdot V_0$  of basis

$V \xrightarrow{w} V'$  does NOT mean " $V' = w(V)$ "!!  
 $L \xrightarrow{s} L'$ ,  $L'$  can be lots of things, not just " $s(L)$ "...

§2 General Case

$G$  a connected reductive group.

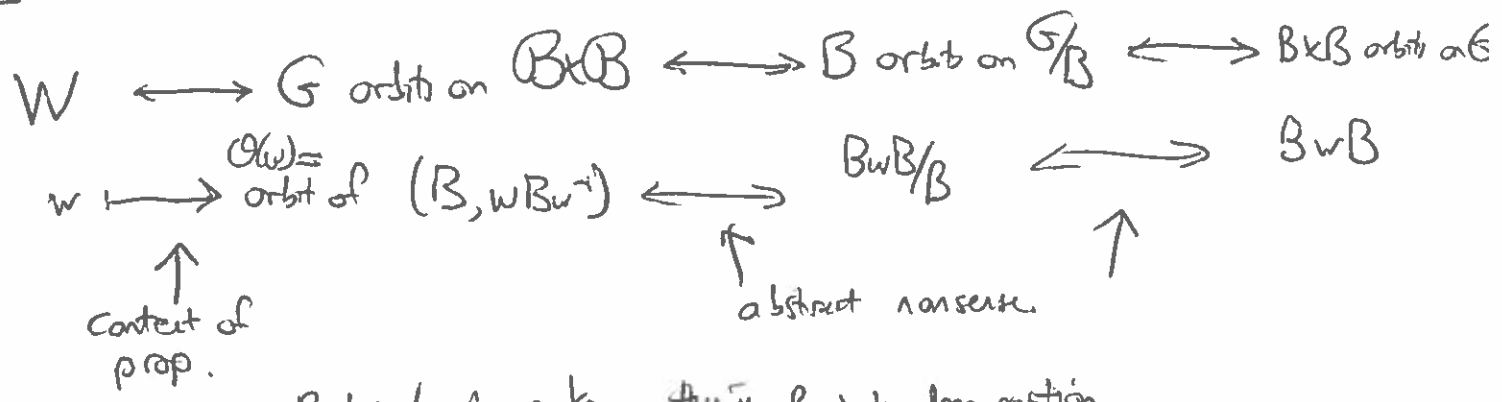
§2.1 Defns, etc

Def: The flag variety of  $G$  is the set  $B$  of all Borel subgroups.

Rank:  $G \curvearrowright B$  by conjugation.  $\text{Stab}_G(B) = N_G(B) = B$  so  $B \cong G/B$   
 $gBg^{-1} \in B$

Like fixing a standard basis, let us fix  $T \subset B$ , and recall  $W = N(T)/T$ .

Prop: The following are in bijection.



Rank: Last we know, thru Bruhat decomposition.

Note: Again,  $W$  does NOT act on  $\mathcal{B}$ . But  $w\mathcal{B}$  (or  $w\mathcal{B}^{-1}$ ) <sup>5/B</sup> <sup>03</sup> (5) still makes sense. For  $w \in \text{NTD}_T$ ,  $f \in \text{WENT}$ , then  $w\mathcal{B}$  makes sense, indep. of IFA since  $T \subset \mathcal{B}$ . [But  $T$  is not in every  $\mathcal{B}$ .]

Def:  $(B_1, B_2)$  in relative position  $w$  if  $(B_1, B_2) \in \mathcal{O}(w)$ , write  $B_1 \xrightarrow{w} B_2$ .

§2.2 [Connection to Coxeter theory]

Key Properties: ①  $\dim_{\mathcal{B} \times \mathcal{B}} \mathcal{O}(w) = \dim \mathcal{B} + l(w) \iff \dim_{\mathcal{B}} \mathcal{B}w\mathcal{B} = l(w)$

② If  $* l(w) + l(w') = l(ww')$  then

$$B_1 \xrightarrow{w} B_2 \xrightarrow{w'} B_3 \implies B_1 \xrightarrow{ww'} B_3$$

Moreover

$$\mathcal{O}(w) \times_{\mathcal{B}} \mathcal{O}(w') \xrightarrow{\sim} \mathcal{O}(ww')$$

is an isomorphism.

$$((B_1, B_2), (B_2, B_3)) \longmapsto (B_1, B_3)$$

↖ ↗  
Same

Link: Need  $*$ ,  
 $L \xrightarrow{s} L' \xrightarrow{s} L''$   
 $\not\Rightarrow L \xrightarrow{1} L''$

We'll explore for  $GL(3)$  in exercises.

Now, an orbit closure is a union of orbits (general nonsense) so

$$\overline{\mathcal{O}(w)} = \bigsqcup_{\text{same } v} \mathcal{O}(v)$$

Def:  $v \leq w$  if  $\mathcal{O}(v) \subseteq \overline{\mathcal{O}(w)}$ .

$$\iff \overline{\mathcal{O}(v)} \subseteq \overline{\mathcal{O}(w)}$$

called the Birkhoff (partial) order.

Ex:  $n=2$ .  $\mathcal{O}(1) = \Delta P^1$  is closed

$$\overline{\mathcal{O}(s)} = P^1 \times P^1 = \mathcal{O}(s) \cup \mathcal{O}(1)$$

$1 \leq s$ .

Lines can "snap together" in closure.

Exer: Agrees w/ "subexpression definition" of Birkhoff order.

Ex: For  $GL(E)$ : closure can make  $\dim(V_s \cap V_{s'})$  increase.

Exer: Verify that  $v \leq w \iff d_{ij}^v \geq d_{ij}^w \forall i, j$ .

Again, we'll explore for  $GL(3)$  in exercises.

(6)

Rank:  $\leq$  has ! minimal element 1.  $O(1) = \Delta B \subset B \times B$  closed.

! maximal element  $w_0$ .  $O(w_0) =$  "generic position"  $\overline{O(w_0)} = B \times B$ .

Rank:  $O(w)$  is an orbit so it is smooth.  $\left\{ \begin{array}{l} \text{Can't be singular everywhere, but} \\ \text{every point in an orbit is the same.} \end{array} \right.$

$\overline{O(w)}$  is typically singular.

Studying singularity of  $\overline{O(w)}$  at a point on  $O(w)$  is (a part of)

Kazhdan-Lusztig theory. A different WARTHO'S - you missed it.