II-2 DELIGNE-LUSZTIG VARIETIES

1) The variety $X(w)$

Recall that $\mathcal{O}(w) \subseteq \mathcal{B}\times\mathcal{B}$ is a G-orbit of dimension $\ell(w) + \dim \mathcal{B}$

def: Given $w \in W$, the Deligne-Lusztig variety $X(w)$ is
$$X(w) := \{(B_1, B_2) \in \mathcal{O}(w) \mid B_2 = F(B_1)\}$$
$$= \mathcal{O}(w) \cap \Gamma_F \leftarrow \text{graph of } F \text{ in } \mathcal{B}\times\mathcal{B}$$

Through the first projection $\mathcal{B}\times\mathcal{B} \to \mathcal{B}$ we get
$$X(w) = \{B \in \mathcal{B} \mid B \xrightarrow{w} F(B)\}$$

Ex: a) Recall that $\mathcal{O}(1) = \Delta\mathcal{B}$

$\rightsquigarrow$ $X(1) = \Delta\mathcal{B}^F \simeq \mathcal{B}^F$ finite set $\simeq (G/B)^F \simeq G^F/B^F$

b) For $G = SL_2$ we have two Deligne-Lusztig varieties

- $X(1) = \mathcal{B}^F = \mathbb{P}_1(\mathbb{F}_q)$
- $X(s) = \mathcal{B} \setminus \mathcal{B}^F = \mathbb{P}_1 \setminus \mathbb{P}_1(\mathbb{F}_q)$

With $\Gamma_F = \{(B, F(B)) \mid B \in \mathcal{B}\} \subseteq \mathcal{B} \times \mathcal{B}$
the variety $X(w)$ is defined via the following cartesian square

$$\begin{tikzcd} X(w) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \Gamma_F \arrow[d, hook] \\ \mathcal{O}(w) \arrow[r, hook] & \mathcal{B} \times \mathcal{B} \end{tikzcd}$$

Since Γ_F is tranverse to $\mathcal{O}(w)$ we deduce :

Prop: $X(w)$ is a smooth quasi-projective variety
of dimension $\ell(w)$

Rmk: $X(w)$ is conjectured to be affine
(proved for $q >$ Coxeter number by Deligne–Lusztig)

The action of G on $\mathcal{O}(w)$ induces an action of the finite reductive group G^F on $X(w)$.

Alternative description

Fix $T \subseteq B$ both F-stable
Then $\mathcal{O}(w) = G \cdot (B, {}^wB) \simeq \{(gB, g'B) \mid g^{-1}g' \in BwB\}$

$\Rightarrow X(w) \simeq \{gB \in G/B \mid g^{-1}F(g) \in BwB\}$

In this description G^F acts by left multiplication on gB
(this does not change $g^{-1}F(g)$)

Recall that $\overline{\mathcal{O}(w)} = \bigsqcup_{v \leq w} \mathcal{O}(v)$

By transversality of Γ_F with any G-orbit on $\mathcal{B} \times \mathcal{B}$
we get

$$\overline{X(w)} = \bigsqcup_{v \leq w} X(v)$$

and $\overline{X(w_0)} \simeq \mathcal{B}$
Note that $\overline{X(w)}$ is smooth whenever $\mathcal{O}(w)$ is.

Prop: If w does not lie in an F-stable parabolic subgroup of W then $X(w)$ is irreducible

For the general case, assume that $I \subseteq S$ is an F-stable set of simple reflections. We can form:

- W_I the parabolic subgroup of W_I
- $P_I = BW_IB$ the parabolic subgroup of G
- $L_I = P_I \cap {}^{w_0}P_I$ the standard Levi subgroup of G

$\rightsquigarrow$ L_I is connected reductive with Weyl group W_I
and $P_I = L_I \ltimes U_I$ with $U_I = R_u(P_I)$
Levi decomposition

Let $w \in W_I$ and $X_{L_I}(w)$ the DL variety in L_I
The action of L_I^F on it can be inflated to an action of P_I^F (with U_I^F acting trivially) and the map

$$\begin{array}{ccc} G^F \times_{P_I^F} X_{L_I}(w) & \longrightarrow & X_G(w) \\ (g, \ell(B \cap L_I)) & \longmapsto & g\ell B \end{array}$$

is a G^F-equivariant isomorphism of varieties.
$\rightsquigarrow$ all the irreducible components of $X(w)$ have dim $\ell(w)$.

2) <u>The variety $\tilde{X}(w)$</u>

Let $U = R_u(B)$ (so that $B = T \ltimes U$)
We replace $\mathcal{B} \simeq G/B$ by $\mathcal{U} \simeq G/U$ and define

$$\tilde{X}(w) = \{ gU \in G/U \mid g^{-1}F(g) \in U\dot{w}U \}$$

need to choose a representative in $N_G(T)$

Again $\tilde{X}(w)$ is smooth of pure dimension $\ell(w)$

G^F acts on $\tilde{X}(w)$ on the left and T^{wF} on the right.

indeed, if $t \in T^{wF}$ then $F(t) = w^{-1} t w$
so that $g^{-1}F(g) \in UwU$

$$\Rightarrow (gt)^{-1} F(gt) = t^{-1} g^{-1} F(g) F(t)$$
$$\in t^{-1} U w U w^{-1} t w = UwU$$

(since T normalizes U)

<u>Prop</u>: The projection $G/U \xrightarrow{/T} G/B$ induces a G^F-equivariant isomorphism of varieties

$$\tilde{X}(w)/T^{wF} \xrightarrow{\sim} X(w)$$

<u>Exercise</u> : $G = SL_2 \supseteq B = \left\{ \begin{pmatrix} \lambda & * \\ \cdot & \lambda^{-1} \end{pmatrix} \right\} \supseteq T = \left\{ \begin{pmatrix} \lambda & \cdot \\ \cdot & \lambda^{-1} \end{pmatrix} \right\}$

1) Show that the maps

$$\begin{array}{ccccc} G & \longrightarrow & \mathbb{A}^2 \setminus \{(0,0)\} & \longrightarrow & \mathbb{P}_1 \\ \begin{pmatrix} a & c \\ b & d \end{pmatrix} & \longmapsto & (a;b) & \longmapsto & [a:b] \end{array}$$

induce G-equivariant isomorphisms $G/U \xrightarrow{\sim} \mathbb{A}^2 \setminus \{(0,0)\}$ and $G/B \xrightarrow{\sim} \mathbb{P}_1$

2) Let $s = \begin{pmatrix} \cdot & 1 \\ -1 & \cdot \end{pmatrix}$. Descibe explicitely UsU and BsB

3) Deduce that $\tilde{X}(s) \simeq \{ (x,y) \in \mathbb{A}_2 \setminus \{(0,0)\} \mid xy^q - yx^q = 1 \}$

and $X(s) \simeq \{ [x:y] \in \mathbb{P}_1 \mid xy^q - yx^q \neq 0 \}$

with the natural map $\tilde{X}(s) \longrightarrow X(s)$.

4) Show that

$$\begin{array}{ccc} \tilde{X}(s) & \longrightarrow & \mathbb{A}_1 \\ (x,y) & \longmapsto & xy^{q^2} - yx^{q^2} \end{array}$$

induces an isomorphism $SL_2(q) \setminus \tilde{X}(s) \xrightarrow{\sim} \mathbb{A}_1$

5) Show that
$$\begin{aligned}\tilde{X}(s) &\longrightarrow \mathbb{A}_1 \setminus \{0\} \\ (x,y) &\longmapsto x\end{aligned}$$
induces an isomorphism $U^F \backslash \tilde{X}(s) \xrightarrow{\sim} \mathbb{A}_1 \setminus \{0\}$

6) Compute $\# \tilde{X}(s)^{tF}$ for any $t \in T^{sF}$