1) The variety $X(w)$

Recall that $O(w) \leq \mathbb{B} \times \mathbb{B}$ is a $G$-orbit of dimension $l(w) + \dim \mathbb{B}$

**Def:** Given $w \in W$, the Deligne-Lusztig variety $X(w)$ is

$$X(w) : = \{ (B_1, B_2) \in O(w) \mid B_2 = F(B_1) \}$$

$$= (O(w) \cap \Gamma_F) \leftarrow \text{graph of } F \text{ in } \mathbb{B} \times \mathbb{B}$$

Through the first projection $\mathbb{B} \times \mathbb{B} \rightarrow \mathbb{B}$ we get

$$X(w) = \{ B \in \mathbb{B} \mid B \overset{w}{\rightarrow} F(B) \}$$

**Ex:**

a) Recall that $O(1) = \Delta \mathbb{B}$

$\Rightarrow$ $X(1) = \Delta \mathbb{B}^F = \mathbb{B}^F$ finite set $\Delta (G/B)^F = G^F/B^F$

b) For $G = SL_2$ we have two Deligne-Lusztig varieties

. $X(1) = \mathbb{B}^F = \mathbb{P}_1(F_q)$

. $X(s) = \mathbb{B} \setminus \mathbb{B}^F = \mathbb{P}_1 \setminus \mathbb{P}_1(F_q)$
With \( \Gamma_F \subset \{(B, F(B)) \mid B \in \mathcal{B}\} \leq \mathcal{B} \times \mathcal{B} \)

the variety \( X(w) \) is defined via the following cartesian square

\[
\begin{array}{ccc}
X(w) & \longrightarrow & \Gamma_F \\
\downarrow & \circlearrowright & \downarrow \\
\mathcal{O}(w) & \longrightarrow & \mathcal{B} \times \mathcal{B}
\end{array}
\]

Since \( \Gamma_F \) is transverse to \( \mathcal{O}(w) \) we deduce:

Prop: \( X(w) \) is a smooth quasi-projective variety

\[ \text{of dimension } l(w) \]

Remk: \( X(w) \) is conjectured to be affine

(proved for \( q > \text{Coxeter number by Deligne-Lusztig} \))

The action of \( G \) on \( \mathcal{O}(w) \) induces an action of the finite reductive group \( G^F \) on \( X(w) \).

**Alternative description**

Fix \( T \subseteq B \) both \( F \)-stable

Then \( \mathcal{O}(w) = G \cdot (B \cdot^w B) \cap \{(gB, g'B) \mid g^{-1}g' \in \mathcal{B} \times \mathcal{B}\} \)

\( \Rightarrow X(w) \cap \{gB \in G/B \mid g^{-1}F(g) \in \mathcal{B} \times \mathcal{B}\} \)
In this description $G$ acts by left multiplication on $gB$ (this does not change $g'$).

Recall that $O(w) = \bigsqcup_{v \leq w} O(v)$.

By transversality of $\Gamma_F$ with any $G$-orbit on $B \times B$ we get

$$X(w) = \bigsqcup_{v \leq w} X(v)$$

and $X(w_0) \cong B$.

Note that $X(w)$ is smooth whenever $O(w)$ is.

Prop: If $w$ does not lie in an $F$-stable parabolic subgroup of $W$ then $X(w)$ is irreducible.

For the general case, assume that $I \subseteq S$ is an $F$-stable set of simple reflections. We can form:

- $W_I$ the parabolic subgroup of $W$
- $P_I = BW_IB$ the parabolic subgroup of $G$
- $L_I = P_I \cap I^o P_I$ the standard Levi subgroup of $G$

$L_I$ is connected reductive with Weyl group $W_I$ and $P_I = L_I \times U_I$ with $U_I = R_u(P_I)$.

$\downarrow$ Levi decomposition
Let $w \in W_I$ and $X_{L_I}(w)$ the DL variety in $L_I$.

The action of $L_I^F$ on it can be inflated to an action of $P_I^F$ (with $U_I^F$ acting trivially) and the map

$$G_X^F \times_{P_I^F} X_{L_I}(w) \to X_G(w)$$

$$(g, l(B_n L_I)) \mapsto glB$$

is a $G^F$-equivariant isomorphism of varieties.

All the irreducible components of $X(w)$ have dim $l(w)$.

2) The variety $\tilde{X}(w)$

Let $U = R_u(B)$ (so that $B = T \times U$).

We replace $B = G/B$ by $U = G/U$ and define

$$\tilde{X}(w) = \{ gU \in G/U \mid g^{-1} F(g) \in U_w U \}$$


Again $\tilde{X}(w)$ is smooth of pure dimension $l(w)$

$G^F$ acts on $\tilde{X}(w)$ on the left and $T^w F$ on the right.
Indeed, if $t \in T^w F$, then $F(t) = w', t w$
so that $g^* F(g) \in U w U$

$\Rightarrow (g t)^* F(g t) = t^* g^* F(g) F(t)$

$\in t^* U w U w^{-1} t w = U w U$

(since $T$ normalizes $U$)

Prop: The projection $G/ U \stackrel{T}{\rightarrow} G/ B$ induces a $G_F$-equivariant isomorphism of varieties

$\tilde{X}(w)/ T^w F \xrightarrow{\sim} X(w)$
Exercise: \( G = Sl_2 \geq B = \left\{ \left( \begin{array}{cc} 1 & \ast \\ \ast & 1 \end{array} \right) \right\} \geq T = \left\{ \left( \begin{array}{cc} 1 & \ast \\ \ast & 1 \end{array} \right) \right\} \)

1) Show that the maps
\[
G \rightarrow A^2 \setminus \{(0,0)\} \rightarrow \mathbb{P}_1, \\
\left( \begin{array}{cc} a & c \\ b & d \end{array} \right) \mapsto (a;b) \mapsto [a:b]
\]
induce \( G \)-equivariant isomorphisms \( G/U \cong A^2 \setminus \{(0,0)\} \) and \( G/B \cong \mathbb{P}_1 \).

2) Let \( s = \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right) \). Describe explicitly \( U s U \) and \( B s B \).

3) Deduce that \( \tilde{X}(s) \cong \left\{ (x,y) \in A^2 \setminus \{(0,0)\} \mid xy^q - yx^q = 1 \right\} \) and \( X(s) \cong \left\{ [x:y] \in \mathbb{P}_1 \mid xy^q - yx^q \neq 0 \right\} \)

with the natural map \( \tilde{X}(s) \rightarrow X(s) \).

4) Show that \( \tilde{X}(s) \rightarrow A_1, \\
(x,y) \mapsto xy^{q^2} - yx^{q^2} \)
induces an isomorphism \( Sl_2(q) \setminus \tilde{X}(s) \cong A_1 \).
5) Show that $(x, y) \mapsto x$ induce an isomorphism $\tilde{X}(s) \cong \mathcal{A}, \setminus \{0\}$;

6) Compute $\# \tilde{X}(s)^{t_F}$ for any $t \in T^{*F}$. 