To $w \in W$ one can attach a Deligne-Lusztig variety $X(w)$
Acts on $X(w)$: 
. the finite reductive group $G^F$
. the finite torus $T^F$ (on $X(w)$)
. the Frobenius $F^\delta$ for any integer $\delta$

\[ F^\delta(w) = w \]

What else?

1) **Braid monoid and braid groups**

Let $(W, S)$ be a (finite) Coxeter system, with presentation

\[ W = \langle S \mid s^2 = 1, sts \ldots = tst \ldots \rangle \]

$m_{st}$ = order of $st$

$m_{st}$ terms

**def:** the Artin-Tits braid monoid $B_w^+$ and braid group $B_w$

are defined by the presentation

\[ B_w^{(+)} = \langle S \mid sts \ldots = tst \ldots \rangle_{gp/\text{monoid}} \]

By definition there is a surjective group/monoid morphism

\[ B_w, B_w^+ \to W \]
It has a set-theoretic splitting \( W \xrightarrow{\beta} B^+_W \) given by reduced expressions. In other words \\
\( \beta(w \cdot w') = \beta(w) \cdot \beta(w') \) if \( l(ww') = l(w) + l(w') \) \\
If there is no ambiguity we will still denote by \( w \) the image \\
of \( w \) by the section \( W \xhookrightarrow{} B^+_W \) \\
The length function on \( W \) extends to \( B^+_W \) and \( B^-_W \) \( (\text{values in } \mathbb{Z}) \)

**Topological construction**

Let \( V \) be a reflection representation of \( W \)
\[ V^w = V \setminus \cup \text{ reflecting hyperplanes} \]
\[ = \text{ subset of } V \text{ where } W \text{ acts freely} \]
Then \( B^+_W \cong \Pi_1(V^w/W, x_0) \)

2) DL varieties attached to els of \( B^+_W \)

\[
\text{def}: \text{ Given } w = (w_1, ..., w_r) \in W \text{ we define } \\
O(w) = O(w_1) \times_{B_1} O(w_2) \times_{B_2} ... \times_{B_{r+1}} O(w_r) \\
= \{(B_1, ..., B_{r+1}) \mid B_1 \xrightarrow{w_1} B_2 \xrightarrow{w_2} ... \xrightarrow{w_r} B_{r+1} \} \\
\text{Then } O(w, w') \subseteq O(ww') \text{ whenever } l(ww') = l(w) + l(w') \]
Moreover this isomorphism is canonical.

One can define $O(b)$ for every $b \in B_w^+$ and a morphism $O(b) \to B \times B$ coming from the first and last projection from $O(w_1, ..., w_r)$.

**Def.** The DL variety $X(b)$ attached to $b \in B_w^+$ is defined by the following cartesian square

\[
\begin{array}{ccc}
X(b) & \xrightarrow{F} & \text{graph of } F \text{ in } B \times B \\
\downarrow & \circlearrowleft & \downarrow \\
O(b) & \to & B \times B
\end{array}
\]

More precisely, if $b = w_1 w_2 ... w_r$ with $w_i \in W$ then

\[
X(b) = X(w_1, ..., w_r) = \{ (B_1, ..., B_r) \mid B_1 \xrightarrow{w_1} B_2 \xrightarrow{w_2} ... \xrightarrow{w_{r-1}} B_r \xrightarrow{w_r} F(B) \}
\]

3) **Cyclic shifts**

If $B_1 \xrightarrow{w} B_2$ then $F(B_1) \xrightarrow{F(w)} F(B_2)$ therefore the Frobenius endomorphism $F$ induces a bijective morphism

\[
\begin{array}{ccc}
X(b) & \to & X(F(b)) \\
(\bar{B}_1, ..., \bar{B}_r) & \mapsto & (F(\bar{B}_1), ..., F(\bar{B}_r))
\end{array}
\]
We can decompose it as follows: if \( b = w_1, w_2, \ldots, w_r \) then we have

\[
\begin{align*}
X(w_1, \ldots, w_r) & \xrightarrow{D_{w_1}} X(w_2, \ldots, w_r, F(w_1)) \xrightarrow{D_{w_2}} X(w_2, \ldots, w_r, F(w_1,w_2)) \ldots \\
(B_1, \ldots, B_r) & \xmapsto{} (B_2, \ldots, B_r, F(B_1)) \xmapsto{} (B_2, \ldots, B_r, F(B_1), F(B_2)) \ldots
\end{align*}
\]

More generally if \( b \in B^+_w \) decomposes as \( b = b_1, b_2 \) with \( b_1, b_2 \in B^+_w \) then \( D_{b_1} : X(b_1, b_2) \xrightarrow{} X(b_2, F(b_1)) \)

\[
\begin{array}{c}
\text{called a cyclic shift operator on } X(b).
\end{array}
\]

**Ex:** \( D_2 = \text{identity} \)

\[
D_b = F \text{ acting on } X(b)
\]

In the particular case where \( b_1, b F(b_1) = b \)

(i.e. when \( b_1 \in C_{B^+_w}(bF) \)) then \( D_{b_1} \in \text{End}(X(b)) \)

Note that \( D_{b_1} \) commutes with the action of \( G^F \)

\[
\Rightarrow \text{ almost an action of } G^F \times C_{B^+_w}(bF) \text{ on } X(b).
\]

(it will be the case on the cohomology of \( X(b) \))