II-3 Braid groups and actions on DL varieties

To $w \in W$ one can attach a Deligne-Lusztig variety $X(w)$

Acts on $X(w)$:
- the finite reductive group G^F
- the finite torus T^{wF} (on $\tilde{X}(w)$)
- the Frobenius F^δ ← any integer δ s.t. $F^\delta(w) = w$

What else?

1) Braid monoid and braid groups

Let (W, S) be a (finite) Coxeter system, with presentation

$$W = \langle S \mid s^2 = 1, \underbrace{sts\cdots}_{m_{st}\text{ terms}} = \underbrace{tst\cdots}_{} \rangle \qquad m_{st} = \text{order of } st$$

def: the Artin-Tits braid monoid B_W^+ and braid group B_W are defined by the presentation

$$B_W^{(+)} = \langle S \mid sts\cdots = tst\cdots \rangle_{gp/(\text{monoid})}$$

By definition there is a surjective group/monoid morphism
$B_W, B_W^+ \twoheadrightarrow W$

It has a set-theoretic splitting $W \xhookrightarrow{\beta} B_W^+$ given by reduced expressions. In other words
$$\beta(w \cdot w') = \beta(w) \cdot \beta(w') \text{ if } \ell(ww') = \ell(w) + \ell(w')$$

If there is no ambiguity we will still denote by w the image of w by the section $W \hookrightarrow B_W^+$

The length function on W extends to B_W^+ and B_W (values in $\mathbb{Z}$)

Topogical construction

Let V be a reflection representation of W
$$V^{reg} = V \setminus \bigcup \text{reflecting hyperplanes}$$
$$= \text{subset of } V \text{ where } W \text{ acts freely}$$
Then $B_W \simeq \pi_1(V^{reg}/W, x_0)$

2) DL varieties attached to elts of B_W^+

def: Given $\underline{w} = (w_1, \ldots, w_r) \in W^r$ we define
$$\mathcal{O}(\underline{w}) = \mathcal{O}(w_1) \times_{\mathcal{B}} \mathcal{O}(w_2) \times_{\mathcal{B}} \cdots \times_{\mathcal{B}} \mathcal{O}(w_r)$$
$$= \left\{ (B_1, \ldots, B_{r+1}) \mid B_1 \xrightarrow{w_1} B_2 \xrightarrow{w_2} \cdots \xrightarrow{w_r} B_{r+1} \right\}$$

Then $\mathcal{O}(w, w') \simeq \mathcal{O}(ww')$ whenever $\ell(ww') = \ell(w) + \ell(w')$

Moreover this isomorphism is canonical

$\rightsquigarrow$ one can define $\mathcal{O}(b)$ for every $b \in B_W^+$ and a morphism $\mathcal{O}(b) \longrightarrow \mathcal{B} \times \mathcal{B}$ coming from the first and last projection from $\mathcal{O}(w_1, \dots, w_r)$

def: The DL variety $X(b)$ attached to $b \in B_W^+$ is defined by the following cartesian square

$$\begin{tikzcd} X(b) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \Gamma_F \arrow[d, hook] \\ \mathcal{O}(b) \arrow[r] & \mathcal{B} \times \mathcal{B} \end{tikzcd}$$

$\longleftarrow$ graph of F in $\mathcal{B} \times \mathcal{B}$

More precisely, if $b = w_1 \cdot w_2 \cdots w_r$ with $w_i \in W$ then

$$X(b) = X(w_1, \dots, w_r) = \left\{ (B_1, \dots, B_r) \,\middle|\, B_1 \xrightarrow{w_1} B_2 \cdots \xrightarrow{w_{r-1}} B_r \xrightarrow{w_r} F(B_1) \right\}$$

3) Cyclic shifts

If $B_1 \xrightarrow{w} B_2$ then $F(B_1) \xrightarrow{F(w)} F(B_2)$ therefore the Frobenius endomorphism F induces a bijective morphism

$$\begin{aligned} X(b) &\longrightarrow X(F(b)) \\ (B_1, \dots, B_r) &\longmapsto (F(B_1), \dots, F(B_r)) \end{aligned}$$

We can decompose it as follows: if $b = w_1 \cdot w_2 \cdots w_r$ then we have

$$X(w_1 \cdots w_r) \xrightarrow{D_{w_1}} X(w_2 \cdots w_r F(w_1)) \xrightarrow{D_{w_2}} X(w_3 \cdots w_r F(w_1 w_2)) \cdots$$
$$(B_1, \ldots, B_r) \longmapsto (B_2, \ldots, B_r, F(B_1)) \longmapsto (B_3, \ldots, B_r, F(B_1), F(B_2)) \cdots$$

More generally if $b \in B_W^+$ decomposes as $b = b_1 \cdot b_2$ with $b_1, b_2 \in B_W^+$ then $D_{b_1} : X(\underbrace{b_1 b_2}_{b}) \longrightarrow X(\underbrace{b_2 F(b_1)}_{b_1^{-1} \cdot b \cdot F(b_1)})$

called a cyclic shift operator on $X(b)$.

Ex: D_1 = identity
D_b = F acting on $X(b)$

In the particular case where $b_1^{-1} b F(b_1) = b$ (i.e when $b_1 \in C_{B_W^+}(bF)$) then $D_{b_1} \in \mathrm{End}(X(b))$

Note that D_{b_1} commutes with the action of G^F $\rightsquigarrow$ almost an action of $G^F \times C_{B_W^+}(bF)$ on $X(b)$.

(it will be the case on the cohomology of $X(b)$)