II-4 THE COXETER VARIETY FOR SL_n

1) The example of SL_2

$$B = \left\{ \begin{pmatrix} \lambda & * \\ \cdot & \lambda^{-1} \end{pmatrix} \right\} \supseteq U = \left\{ \begin{pmatrix} 1 & * \\ \cdot & 1 \end{pmatrix} \right\} \quad \text{and} \quad s = \begin{pmatrix} \cdot & 1 \\ -1 & \cdot \end{pmatrix}$$

We have seen that the isomorphisms

$$\begin{aligned} \mathbb{A}_2 \setminus \{(0,0)\} &\xrightarrow{\sim} G/U \\ (x,y) &\longmapsto \begin{pmatrix} x & * \\ y & * \end{pmatrix} U \end{aligned}$$

and
$$\begin{aligned} \mathbb{P}_1 &\xrightarrow{\sim} G/B \\ [x:y] &\longmapsto \begin{pmatrix} x & * \\ y & * \end{pmatrix} B \end{aligned}$$

induce
$$\begin{aligned} &\tilde{X}(s) \simeq \{(x,y) \in \mathbb{A}_2 \mid xy^q - yx^q = 1\} \\ &\quad \downarrow / \mu_{q+1} \\ &X(s) \simeq \{[x:y] \in \mathbb{P}_1 \mid xy^q - yx^q \neq 0\} = \mathbb{P}_1 \setminus \mathbb{P}_1(\mathbb{F}_q) \end{aligned}$$

In addition the map
$$\begin{aligned} X(s) &\longrightarrow \mathbb{G}_m \\ [x:y] &\longmapsto \left(\frac{x}{y}\right)^q - \frac{x}{y} \end{aligned}$$

induces an isomorphism $\quad U^F \backslash X(s) \xrightarrow{\sim} \mathbb{G}_m$

2) The variety $X(w)$ for $w = (1,2,\dots,n)$

Recall that for $G = Gl_n$ or Sl_n, elt's of $\mathcal{B}$ are flags of vector spaces $V_\bullet = (\{0\} = V_0 \subseteq V_1 \subseteq \dots \subseteq V_n = \overline{\mathbb{F}}_p^{\,n})$

Prop: The map

$$\begin{aligned} \mathcal{B} &\longrightarrow \mathbb{P}(\overline{\mathbb{F}}_q^{\,n}) \\ V_\bullet &\longmapsto V_1 \end{aligned}$$

induces an isomorphism

$$X((1,2,\dots,n)) \xrightarrow{\sim} \{ L \in \mathbb{P}(\overline{\mathbb{F}}_q^{\,n}) \mid L + F(L) + \dots + F^{n-1}(L) = \overline{\mathbb{F}}_p^{\,n} \}$$

proof: $X((1,2,\dots,n)) = \{ V_\bullet \in \mathcal{B} \mid V_\bullet \xrightarrow{w} F(V_\bullet) \}$

Let $e_1, \dots, e_n$ be a basis adapted to $V_\bullet$ s.t
$e_{w(1)}, \dots, e_{w(n)}$ $(= e_2, \dots, e_n, e_1)$ is adapted to $F(V_\bullet)$

Then $F(V_1) = \langle e_2 \rangle \implies V_2 = \langle e_1, e_2 \rangle = V_1 \oplus F(V_1)$

$F(V_2) = \langle e_2, e_3 \rangle \implies V_3 = V_1 \oplus F(V_2)$

$= V_1 \oplus (F(V_1) \oplus F^2(V_1))$

.... until $V_n = \overline{\mathbb{F}}_q^{\,n} = V_1 \oplus F(V_1) \oplus \dots \oplus F^{n-1}(V_1)$

Conversely, if $L \in \mathbb{P}(\overline{\mathbb{F}}_q^{\,n})$ is s.t $L + F(L) + \dots + F^{n-1}(L) = \overline{\mathbb{F}}_q^{\,n}$
then $L + F(L) + \dots + F^{i-1}(L)$ has dimension i and defines a flag $V_\bullet = (\{0\} \subseteq L \subseteq L \oplus F(L) \cdots)$ such that $V_\bullet \xrightarrow{w} F(V_\bullet)$ $\square$

Using coordinates we get

$$X((1,2,\dots,n)) \simeq \left\{ [x_1 : \dots : x_n] \in \mathbb{P}_{n-1}(\overline{\mathbb{F}}_p) \;\middle|\; \det \underbrace{\begin{pmatrix} x_1 & x_1^q & & x_1^{q^{n-1}} \\ \vdots & \vdots & \dots & \vdots \\ x_n & x_n^q & & x_n^{q^{n-1}} \end{pmatrix}}_{\Delta(x_1,\dots,x_n)} \neq 0 \right\}$$

$(1,2,\dots,n)$ is a Coxeter elt of $\mathfrak{S}_n$ and the Deligne-Lusztig var.
$X_n = X((1,\dots,n))$ is called a Coxeter variety

Prop: (i) X_n is an affine variety of dim $n-1$

(ii) $$\Delta(x_1,\dots,x_n) = (-1)^{\binom{n}{2}} \prod_{i=1}^{n} \prod_{a_{i+1},\dots,a_n \in \mathbb{F}_q} (x_i + a_{i+1}x_{i+1} + \dots + a_n x_n)$$

therefore $X_n = \mathbb{P}_{n-1} \setminus$ rational hyperplanes

(iii) $X_n^{F^i} = \emptyset$ for $i = 1, \dots, n-1$

proof: $x_n = 0 \Rightarrow \Delta(x_1,\dots,x_n) = 0$ and Δ is homogenous
therefore
$$X_n \simeq \{ (x_1,\dots,x_{n-1}) \in \mathbb{A}_{n-1} \mid \Delta(x_1,\dots,x_{n-1},1) \neq 0 \}$$
which proves (i)

For (ii) we observe that if $a_1 x_1 + \dots + a_n x_n = 0$
for some a_i's in $\mathbb{F}_q$ not all zero then $\Delta(x_1,\dots,x_n) = 0$

Since Δ is homogenous of degree $1+\dots+q^{n-1} = \# \mathbb{P}^{\vee}_{n-1}(\mathbb{F}_q)$ and vanishes on every rational hyperplanes, we deduce (ii) up to some scalar.

(iii) is straightforward since $L \cap F^i(L) = 0$ for every $L \in X_n$ and $1 \leqslant i \leqslant n-1$ $\square$

$\underline{\text{Ex}}$: $n=2$ $\begin{vmatrix} x & x^q \\ y & y^q \end{vmatrix} = xy^q - yx^q$

3) <u>The variety $\tilde{X}(w)$</u>

The n-cycle w can be lifted in SL_n as $\begin{bmatrix} 0 & & & (-1)^{n-1} \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ (0) & & 1 & 0 \end{bmatrix}$

and we can consider $\tilde{X}_n = \tilde{X}(w)$ for this representative in $N_G(T)$

Prop: The map $gU \in G/U \mapsto g(e_1)$ induces an isomorphism
$$\tilde{X}_n \simeq \{ (x_1, \dots, x_n) \in \mathbb{A}_n \mid \Delta(x_1, \dots, x_n) = 1 \}$$

Recall that $\tilde{X}(w) \twoheadrightarrow X(w)$ is the quotient by T^{wF}
$$gU \longmapsto gB$$

Here $T^{wF} = \left\{ \begin{pmatrix} \lambda & & & \\ & \lambda^q & & \\ & & \ddots & \\ & & & \lambda^{q^{n-1}} \end{pmatrix} \mid \lambda \in \mu_{1+\dots+q^{n-1}}(\mathbb{F}_q) \right\}$

So that this quotient map becomes

$$\begin{array}{c} \tilde{X}_n = \{ (x_1,\dots,x_n) \in \mathbb{A}_n \mid \Delta(x_1,\dots,x_n) = 1 \} \\ \big\downarrow \; / \mu_{1+q+\dots+q^{n-1}} \leftarrow \text{homogenous degree of } \Delta \\ X_n = \{ [x_1 : \dots : x_n] \in \mathbb{P}_{n-1} \mid \Delta(x_1,\dots,x_n) \neq 0 \} \end{array}$$

With $n=2$ we recover the case of the Drinfeld curve