Computing cohomology groups by counting points
WARTHOG 2018

Fix: • prime power q
• prime ℓ with $\ell \nmid q$.
• quasiprojective variety X over $\overline{\mathbb{F}}_q$.

$\rightsquigarrow$ ℓ-adic étale cohomology groups $H^i(X)$, $H^i_c(X)$
(vector spaces over $\mathbb{Q}_\ell$).

I won't define these groups, but I will tell you all of the properties you need to know, and use these properties to compute some nontrivial examples.

Properties:

① Functoriality: $f: X \to Y$ induces $f^*: H^i(Y) \to H^i(X)$.
If f is proper, also get $f_*: H^i_c(X) \to H^i_c(Y)$
$\rightsquigarrow$ If $G \circlearrowright X$, then $G \circlearrowright H^i(X)$ and $G \circlearrowright H^i_c(X)$.

② Poincaré duality: If X is smooth and connected of dimension n, $\exists$ perfect pairing $H^i(X) \otimes H^{2n-i}_c(X) \to \mathbb{Q}_\ell$

③ Always have $H^i_c(X) \to H^i(X)$, and it's an isomorphism if X is projective.

4 $H^i(\mathbb{A}^n) = \begin{cases} \mathbb{Q}_\ell & \text{if } i=0 \\ 0 & \text{otherwise} \end{cases}$

$H^i_c(\mathbb{A}^n) = \begin{cases} \mathbb{Q}_\ell & \text{if } i=2n \\ 0 & \text{otherwise} \end{cases}$

Note how this fits with properties 2 and 3

5 Long exact sequence: $U \subset X$ open $\rightsquigarrow$

$$\cdots \to H^i_c(U) \to H^i_c(X) \to H^i_c(X \setminus U) \to H^{i+1}_c(U) \to \cdots$$

The last property that I want to state is by far the most complicated, but also the most important for this talk.

6 Suppose we have a Frobenius endomorphism $F \circlearrowright X$, ie X is defined by equalities and inequalities with coefficients in $\mathbb{F}_q \subset \overline{\mathbb{F}}_q$

$\rightsquigarrow Fr_q \circlearrowright H^i(X)$ and $H^i_c(X)$

a) 1, 3, and 5 are equivariant.

b) 2 is equivariant, where $F_q \circlearrowright \mathbb{Q}_\ell$ as mult. by q^n

c) $F_q \circlearrowright H^0(\mathbb{A}^n)$ trivially
$F_q \circlearrowright H^{2n}_c(\mathbb{A}^n)$ as mult by q^n $\Big\}$ equivalent by (b)

d) Lefschetz fixed point formula:

$$|X^F_{\mathbb{F}_q}| = \sum_i (-1)^i \, tr(F \circlearrowright H^i_c(X))$$

Now we're ready to do some computations. Let's start by computing the cohomology of $\mathbb{P}^n$. This can certainly be done directly using 5, but it's fun to use 6, and it will be a good warm up for some important examples down the road.

Lemma 1: $H^{odd}(\mathbb{P}^n) = 0$ and $F_{\mathbb{F}_q} \circlearrowright H^{2i}(\mathbb{P}^n)$ as mult by q^i

Pf by induction: $\mathbb{P}^n = \mathbb{P}^{n-1} \amalg \mathbb{A}^n$. Use LES (exercise).

Lemma 2: Let $f_n(t) = 1 + t + \dots + t^n$. $\forall s$, $|\mathbb{P}^n_{\mathbb{F}_{q^s}}| = f_n(q^s)$

Pf: $\mathbb{P}^n = \mathbb{P}^{n-1} \amalg \mathbb{A}^n$

Prop: $\sum_i \dim H^{2i}(\mathbb{P}^n)\, t^i = f_n(t)$

Pf: By Lefschetz, $f_n(q^s) = |\mathbb{P}^n_{\mathbb{F}_{q^s}}| = |(\mathbb{P}^n)^{F^s}|$

$$f_n(q^s) = |(\mathbb{P}^n)^{F^s}| = \sum_i tr(F^s \circlearrowright H^{2i}(\mathbb{P}^n)) = \sum_i [\text{illegible}] \dim H^{2i}(\mathbb{P}^n) = \sum_i \dim H^{2i}(\mathbb{P}^n)\, q^{si}$$

[illegible] $H^* = H^*$

Since these two polynomials agree at infinitely many points, they must be the same polynomial.

Now let's do something similar for complements of hyperplane arrangements.

Let $\mathcal{A}$ be a finite set of q-rational hyperplanes in $\mathbb{A}^n_{\mathbb{F}_q}$ (not necessarily through the origin).

Let $M_{\mathcal{A}} = \mathbb{A}^n_{\mathbb{F}_q} \setminus \bigcup_{H \in \mathcal{A}} H_{\mathbb{F}_q}$

Lemma 1': $H_c^{2n-i}(M_{\mathcal{A}}) = 0$ unless $0 \le i \le n$, and $\mathrm{Fr} \circlearrowright H_c^{2n-i}(M_{\mathcal{A}})$ as mult by q^{n-i}.

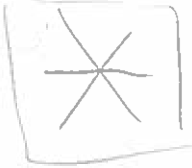
Pf: Induction on $|\mathcal{A}|$. Fix $H_0 \in \mathcal{A}$. Let

$$\mathcal{A}' = \mathcal{A} \setminus \{H_0\}, \quad \mathcal{A}'' = \{H \cap H_0 \mid H_0 \neq H \in \mathcal{A}\}$$

Ex:

$M_{\mathcal{A}'} = M_{\mathcal{A}} \amalg M_{\mathcal{A}''}$. Use LES (exercise).

Lemma 2′: $\exists$ a monic polynomial $\chi_A(t)$ of degree n st $\forall s$, $|(M_A)_{\mathbb{F}_{q^s}}| = \chi_A(q^s)$.

Ex:  $\chi_A(t) = t^2 - 3t + 2$.

Pf: $\chi_A(t) = \underbrace{\chi_{A'}(t)}_{\text{monic of degree } n} - \underbrace{\chi_{A''}(t)}_{\text{degree } n-1}$

Just like we did with $\mathbb{P}^n$, we can combine these lemmas to describe the Poincaré polynomial.

Lefschetz: $\chi_A(q^s) = |(M_A)_{\mathbb{F}_{q^s}}| = |M_A^{F^s}|$

$$= \sum_i (-1)^i \operatorname{tr}(F^s \circlearrowright H^i_c(M_A))$$

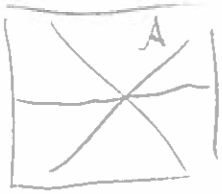
$$= \sum_{i=0}^{n} (-1)^{2n-i} \operatorname{tr}(F^s \circlearrowright H^{2n-i}_c(M_A))$$

$$= \sum_{i=0}^{n} (-1)^i (q^s)^{n-i} \dim H^{2n-i}_c(M_A)$$

So $\chi_A(t) = \sum_{i=0}^{n} (-1)^i t^{n-i} \dim H^{2n-i}_c(M_A)$

$$= \sum^{n} (-1)^i t^{n-i} \dim H^i(M_A)$$

Totally different but equally useful relationship between Poincaré poly and point counting

Ex: 

$$H^i(M_A) = \begin{cases} \mathbb{Q}_\ell, & i=0 \\ \mathbb{Q}_\ell^3, & i=1 \\ \mathbb{Q}_\ell^2, & i=2. \end{cases}$$

Ex: A = all q-rational hyperplanes in $\mathbb{A}^{n-1}$

$\rightsquigarrow M_A = X_n$ (SL_n Coxeter variety)

- $\chi_A(t)$ monic of degree $n-1$
- We've seen that $\chi_A(q) = \chi_A(q^2) = \dots = \chi_A(q^{n-1}) = 0$

$$\Rightarrow \chi_A(t) = (t-q)(t-q^2)\cdots(t-q^{n-1}).$$

$$\| $$

$$\sum_{i=0}^{n-1} (-1)^i t^{n-1-i} \dim H^i(X_n)$$