Computing cohomology groups by counting points

WARTHOA 2018

Fix a prime power $q$
- prime $l$ with $l \nmid q$.
- quasi-projective variety $X$ over $\overline{\mathbb{F}}_q$.

$L$-adic étale cohomology groups
$H^i(X), H^i_c(X)$ (vector spaces over $\mathbb{Q}_l$).

I won't define these groups, but I will tell you all of the properties you need to know, and use these properties to compute some nontrivial examples.

Properties:

1. Functoriality: $f: X \to Y$ induces $f^*: H^i(Y) \to H^i(X)$.
   - If $f$ is proper, also get $f_!: H^i_c(X) \to H^i_c(Y)$
   - If $C \subset X$, then $C \cap H^i_c(X)$ and $C \cap H^i_c(X)$.

2. Poincaré duality: if $X$ is smooth and connected of dimension $n$, there is a perfect pairing $H^i(X) \otimes H^{2n-i}_c(X) \to \mathbb{Q}_l$.

3. Always have $H^i_c(X) \to H^i(X)$, and it's an isomorphism if $X$ is projective.
4. \( H^i(A^n) = \begin{cases} \mathbb{Q} & \text{if } i = 0 \\ 0 & \text{otherwise} \end{cases} \)

\( H^i_c(A^n) = \begin{cases} \mathbb{Q} & \text{if } i = 2n \\ 0 & \text{otherwise} \end{cases} \)

Note how this fits with properties 2 and 3.

5. Long exact sequence: \( U \subset X \) open and 

\[ \cdots \to H^i_c(U) \to H^i_c(X) \to H^i_c(X \cup U) \to H^{i+1}_c(U) \to \cdots \]

The last property that I want to state is by far the most complicated, but also the most important for this talk.

6. Suppose we have a Frobenius endomorphism \( F: X \to X \), i.e., \( X \) is defined by equalities and inequalities, with coefficients in \( \mathbb{F}_q \subset \overline{\mathbb{F}_q} \).

\[ F: \mathbb{Q} \cap H^i(X) \text{ and } H^i_c(X) \]

a) 0, 2, and 5 are equivariant.

b) \( 2 \) is equivariant, where \( F: \mathbb{Q} \cap \text{mult. by } q^n \)

c) \( F: \mathbb{Q} \cap H^0(A^n) \) trivially

\( F: \mathbb{Q} \cap H^{2n}_c(A^n) \) as mult. by \( q^n \) \( \Rightarrow \) equivalent by (b)
\( d) \) Lefschetz fixed point formula:
\[
|X| = \sum_i (-1)^i \text{tr}(F^i \circ H^i(X))
\]

Now we're ready to do some computations. Let's start by computing the cohomology of \( \mathbb{P}^n \). This can certainly be done directly using (5), but it's fun to use (6), and it will be a good warm-up for some important examples down the road.

**Lemma 1:** \( H^{2i}(\mathbb{P}^n) = 0 \) and \( F^i \in H^{2i}(\mathbb{P}^n) \) as mult by \( q^i \)

*Proof by induction:* \( \mathbb{P}^n = \mathbb{P}^{n-1} \cup A^n \). Use LES (exercise).

**Lemma 2:** Let \( f_n(t) = 1 + t + \ldots + t^n \). For all \( s \), \( |\mathbb{P}^n_{F_2^s}| = f_n(q^s) \)

*Proof:* \( \mathbb{P}^n = \mathbb{P}^{n-1} \cup A^n \)

**Prop.:** \( \sum_i \dim H^{2i}(\mathbb{P}^n) t^i = f_n(t) \)

*Proof:* By Lefschetz, \( f_n(q^s) = |\mathbb{P}^n_{F_2^s}| = |\mathbb{P}^n| q^s \)

\[
|\mathbb{P}^n | = \sum_i \text{tr}(F^i \circ H^i(\mathbb{P}^n)) = \sum_i \dim H^{2i}(\mathbb{P}^n) q^i
\]
Since these two polynomials agree at infinitely many points, they must be the same polynomial.

Now let's do something similar for complements of hyperplane arrangements.

Let $A$ be a finite set of $\mathbb{A}^n$-hyperplanes in $\mathbb{A}^n$. Let $M_A = \mathbb{A}^n \setminus \cup_{H \in A} H$. 

Lemma 1': $H^*_c(M_A) = 0$ unless $0 \leq i \leq n$, and $\text{Fr}_c \cap H^{2n-i}_c(M_A)$ as mult by $q^{n-i}$.

Pf: Induction on $|A|$. Fix $H_0 \in A$. Let $A' = A \setminus \{H_0\}, \ A'' = \{H_0 \cap H \mid H_0 \neq H \in A\}$.

Ex: \[ \begin{array}{c}
\text{Ex:} & \quad \begin{array}{c}
\begin{array}{c}
H_0 \ A
\end{array}
\end{array}
\end{array}\]

$M_A' = M_A \cup M_A''$. Use LES (exercise).
Lemma 2': \( \exists \) a monic polynomial \( \chi_A(t) \) of degree \( n \) such that \( \forall s, \ |(M_A)_{F_{q^s}}| = \chi_A(q^s) \).

Example: \( \chi_A(t) = t^2 - 3t + 2 \).

Proof: \( \chi_A(t) = \chi_A(t) - \chi_A''(t) \), monic of degree \( n-1 \).

Just like we did with \( \rho^* \), we can combine these lemmas to describe the Poincaré polynomial.

\[
\text{Lefschetz: } \chi_A(q^s) = |M_A|_{F_{q^s}} = \prod_{i=1}^{\dim M_A} (1 - q^s t^i)
\]

\[
= \sum (-1)^i \text{tr} (F_{q^s}^i \cdot H_c^i(M_A))
\]

\[
= \sum (-1)^{2n-i} \text{tr} (F_{q^s}^i \cdot H_c^{2n-i}(M_A))
\]

\[
= \sum (-1)^i (q^s)^{n-i} \dim H_c^{2n-i}(M_A)
\]

So \( \chi_A(t) = \sum (-1)^i t^{n-i} \dim H_c^{2n-i}(M_A) \), Totally different but equally useful relationship between Poincaré poly and point counting.
Ex: \( A \) is all hyperplanes in \( \mathbb{A}^{n-1} \)
and \( M_A = X_n \) (SL_n Cameron variety)

- \( X_A(t) \) monic of degree \( n-1 \)

- We've seen that \( X_A(q) = X_A(q^2) = \ldots = X_A(q^{n-1}) = 0 \)

\[ \Rightarrow X_A(t) = (t-q)(t-q^2) \ldots (t-q^{n-1}). \]

\[ \sum_{i=0}^{n-1} (-1)^i t^{n-1-i} \dim H^i(X_n) \]