1) Action of a finite group

Let \( H \) be a finite group acting by automorphisms on \( X \). The linear action of \( H \) on \( H^i_c(X) \) by functionality gives the contribution of the trivial representation in \( H^i_c(X) \).

\[
\text{Prop: } H^i_c(H \setminus X) \cong H^i_c(X) \cong \mathbb{Q}_\ell \otimes_{\mathbb{Q}_\ell H} H^i_c(X)
\]

This gives the contribution of the trivial representation in \( H^i_c(X) \).

\[
\text{Ex: } \tilde{X}(w) = \{ gU \in G/U \mid g^*F(g) \in UwU \}
\]

\[
X := \{ g \in G \mid g^*F(g) \in UwU \}
\]

The fibers are affine spaces isomorphic to \( U \).

\[
\Rightarrow H^i_c(\tilde{X}(w)) = H^{i+2\dim U}_c(X)
\]

Now \( X \rightarrow UwU \) induces \( X \cong UwU \)

\[
g \mapsto g^*F(g)
\]
\[ H_c^* (G^F \setminus X) \cong \overline{Q}_\ell [-2 \dim X] \]

**Corollary:** the trivial representation of \( G^F \) occurs

only in the top degree of \( H_c^* (\tilde{X}(w)) \)

and with multiplicity one

**Rmk:** if \( X, Y \) are both endowed with an action of \( H \)

we can form the amalgamated product \( X \times_H Y = (X \times Y) / \Delta H \)

\[ \Rightarrow H_c^* (X \times_H Y) \cong H_c^* (X) \otimes_{\overline{Q}_\ell H} H_c^* (Y) \]

**Ex:** if \( w \in W_I \) the isomorphism \( G^F_{x_{P}^F} \times_{P^F} X_{L_I}(w) \cong X(w) \)

gives

\[ H_c^* (X(w)) = \overline{Q}_\ell G^F \otimes_{\overline{Q}_\ell P^F_I} H_c^* (X_{L_I}(w)) \]

\[ = \text{Ind}^{G^F}_{P^F_I} (\text{Inf}_{L^F_I}^{P^F_I} H_c^* (X_{L_I}(w))) \]

Haish-Chandra induction

**Prop:** If the action of \( H \) extends to an action of \( \Gamma \)

a connected algebraic group then \( H_c^* (X) \wedge H_c^* (X)^H \)

such actions are "boring".
Ex: \( X(\omega_0) \to \mathbb{B} \) the action of \( G^F \) extends to \( G \)
\[ \implies \text{only the trivial representation occurs in } H^*(X(\omega_0)) \]

2) **Convergence of the trace formula**

**Thm (Lefschetz trace formula)**
\[ \# X^F = \sum_{i \in \mathbb{Z}} (-1)^i \text{Trace}(F|H^i_c(X)) \]

Assume now that we have both

- a Frobenius \( F: X \to X \) for some \( \mathbb{F}_q \)-structure
- an action of the finite group \( H \) on \( X \)

such that \( F^t(h.x) = h \cdot F^t(x) \quad \forall h \in H, x \in X \)

We can define the class function
\[ L: h \mapsto \sum_{i \in \mathbb{Z}} (-1)^i \text{Trace}(h|H^i_c(X)) \]

it is a character of a virtual representation, hence
\( L(h) \) is an algebraic integer for all \( h \in H \)

**Prop:** \( L(h) = \lim_{t \to \infty} \sum_{n \geq 1} \# X^{hF^n} t^n \)
proof: \( hF = Fh \) so we can simultaneously triangulate with eigenvalues \( \lambda_{i,j}, \ldots, \lambda_{i,r} \) of \( h \) on \( H_c^i(X) \)
\( \mu_{i,j}, \ldots, \mu_{i,r} \) of \( F \)

Since \( hF^n \) is a Frobenius endomorphism of \( X \)
\[
\#X^{hF^n} = \text{Trace}(hF^n|H_c^i(X))
\]
\[
= \sum_{i \in \mathbb{Z}} (-1)^i \sum_j \lambda_{i,j} \mu_{i,j}^n
\]

\[
\Rightarrow \sum_{n \geq 1} \#X^{hF^n} t^n = \sum_{i,j} (-1)^i \lambda_{i,j} \sum_{n \geq 1} \left( \frac{\mu_{i,j} t}{1 - \mu_{i,j} t} \right)^n
\]

\[
\frac{\mu_{i,j} t}{1 - \mu_{i,j} t} \to -1 \quad \text{as} \quad t \to \infty
\]

and \( \mathcal{L}(h) = \sum_{i,j} (-1)^i \lambda_{i,j} \)

Corollary: \( \mathcal{L}(h) = \sum (-1)^i H_c^i(X) \) is an integer
\( \frac{1}{\text{independent of } \ell} \)

proof: \( \mathcal{L}(h) \) is an algebraic integer which is a limit of a formal series with integral coefficients independent of \( \ell \) \( \square \)
3. Action of $p'$-elements

Feature from the cohomology over $\mathbb{F}_p, \mathbb{Z}_p$.

$\ell \cdot |\text{Stab}_H(x)|$ for all $x \in X$

$\Rightarrow \mathcal{L}$ is the lift of a virtual projective representation.

Corollary: If $h$ is a $p'$-elt then $\mathcal{L}_x(h)$ equals the Euler char. of $X^h$

Proof: Assume first that $h$ order $r$ for some prime number. Since $\mathcal{L}(h)$ does not depend on $\ell$

one can assume $r = \ell$

Then every projective representation of $\langle h \rangle$ is multiple of the regular rep., on which $\text{Trace}(h)$ is zero.

Since $\langle h \rangle$ acts freely on $X \backslash X^h = X \backslash X^{\langle h \rangle}$

we deduce that

$\sum (\epsilon_i) \text{Trace}(h \mid H^\ast_c(X \backslash X^h)) = 0$

from which we get

$\mathcal{L}(h) = \sum (\epsilon_i) \text{Trace}(h \mid H^\ast_c(X^h))$

$= \sum (\epsilon_i) \dim H^\ast_c(X^h)$

For the general case we decompose $h$ and use induction. ∎