III-2 GROUP ACTIONS AND TRACE FORMULA

1) Action of a finite group

Let H be a finite group acting by automorphisms on X
$\rightsquigarrow$ linear action of H on $H^i_c(X)$ by functoriality

Prop: $H^i_c({}_H\backslash X) \simeq \underbrace{H^i_c(X)^H}_{\text{invariants}} \simeq \underbrace{\overline{\mathbb{Q}}_\ell \otimes_{\overline{\mathbb{Q}}_\ell H} H^i_c(X)}_{\text{coinvariants}}$

$\rightsquigarrow$ gives the contribution of the trivial representation in $H^\bullet_c(X)$

Ex: $\tilde{X}(w) = \{ gU \in G/U \mid g^{-1}F(g) \in UwU \}$

$$\uparrow$$

$X := \{ g \in G \mid g^{-1}F(g) \in UwU \}$

The fibers are affine spaces isomorphic to U
$\Rightarrow H^i_c(\tilde{X}(w)) \simeq H^{i+2\dim U}_c(X)$

Now $X \twoheadrightarrow UwU$, $g \mapsto g^{-1}F(g)$ induces ${}_{G^F}\backslash X \xrightarrow{\sim} UwU$

$$\Longrightarrow \quad H^\bullet_c(G^F\backslash X) \simeq \overline{\mathbb{Q}}_\ell[-2\dim X]$$

Corollary: the trivial representation of G^F occurs only in the top degree of $H^\bullet_c(\tilde{X}(w))$ and with multiplicity one

Rmk: if X, Y are both endowed with an action of H we can form the amalgamated product $X \times_H Y = (X \times Y)/\Delta H$

$$\rightsquigarrow \quad H^\bullet_c(X \times_H Y) \simeq H^\bullet_c(X) \otimes_{\overline{\mathbb{Q}}_\ell H} H^\bullet_c(Y)$$

Ex: if $w \in W_I$ the isomorphism $G^F \times_{P_I^F} X_{L_I}(w) \xrightarrow{\sim} X(w)$ gives

$$H^\bullet_c(X(w)) = \overline{\mathbb{Q}}_\ell G^F \otimes_{\overline{\mathbb{Q}}_\ell P_I^F} H^\bullet_c(X_{L_I}(w))$$

$$= \underbrace{\mathrm{Ind}_{P_I^F}^{G^F}\left(\mathrm{Inf}_{L_I^F}^{P_I^F}\right.}_{\text{Harish-Chandra induction}} \left. H^\bullet_c(X_{L_I}(w))\right)$$

Prop: If the action of H extends to an action of a connected algebraic group then $H^\bullet_c(X) \simeq H^\bullet_c(X)^H$

$\rightsquigarrow$ such actions are "boring".

Ex: $\overline{X(w_0)} \simeq \mathcal{B}$ the action of G^F extends to G
$\Rightarrow$ only the trivial representation occurs in $H^\bullet(\overline{X(w_0)})$

2) Consequence of the trace formula

Thm (Lefschetz trace formula)
$$\# X^F = \sum_{i\in\mathbb{Z}} (-1)^i \, \mathrm{Trace}(F \mid H^i_c(X))$$

Assume now that we have both
- a Frobenius $F : X \longrightarrow X$ for some $\mathbb{F}_q$-structure
- an action of the finite group H on X

such that $F(h\cdot x) = h\cdot F(x) \quad \forall h\in H, x\in X$

We can define the class function

$$\mathcal{L} : h \longmapsto \sum_{i\in\mathbb{Z}} (-1)^i \, \mathrm{Trace}(h \mid H^i_c(X))$$

it is a character of a virtual representation hence $\mathcal{L}(h)$ is an algebraic integer for all $h\in H$

Prop: $\mathcal{L}(h) = -\lim\limits_{t\to\infty} \sum\limits_{n\geq 1} \# X^{hF^n} t^n$

proof: $hF = Fh$ so we can simultaneously triangularize with eigenvalues $\lambda_{i,1}, \dots, \lambda_{i,r}$ of h on $H^i_c(X)$
$\mu_{i,1}, \dots, \mu_{i,r}$ of F

Since hF^n is a Frobenius endomorphism of X

$$\# X^{hF^n} = \mathrm{Trace}(hF^n \mid H^\bullet_c(X))$$

$$= \sum_{i \in \mathbb{Z}} (-1)^i \sum_j \lambda_{i,j} \mu_{ij}^n$$

$$\Rightarrow \sum_{n \geq 1} \# X^{hF^n} t^n = \sum_{i,j} (-1)^i \lambda_{ij} \underbrace{\sum_{n \geq 1} (\mu_{ij} t)^n}_{\frac{\mu_{ij} t}{1 - \mu_{ij} t} \xrightarrow[t \to \infty]{} -1}$$

and $\mathcal{L}(h) = \sum_{i,j} (-1)^i \lambda_{ij}$

Corollary: $\mathcal{L}(h) = \sum (-1)^i H^i_c(X)$ is an integer independent of ℓ

proof: $\mathcal{L}(h)$ is an algebraic integer with is a limit of a formal series with integral coefficients indep / ℓ □

3- Action of p'-elements

Feature from the cohomology over $\mathbb{F}_\ell, \mathbb{Z}_\ell$:

$\ell \nmid |\mathrm{Stab}_H(x)|$ for all $x \in X$

$\Rightarrow$ $\mathcal{L}$ is the lift of a virtual projective representation

Corollary : If h is a p'-elt then $\mathcal{L}_X(h)$ equals the Euler char. of X^h

proof: assume first that h order r for r some prime number. Since $\mathcal{L}(h)$ does not depend on ℓ one can assume $r=\ell$

Then every projective representation of $\langle h \rangle$ is multiple of the regular rep., on which Trace(h) is zero

Since $\langle h \rangle$ acts freely on $X \setminus X^h = X \setminus X^{\langle h \rangle}$ we deduce that

$$\sum (-1)^i \,\mathrm{Trace}(h \mid H^i_c(X \setminus X^h)) = 0$$

from which we get

$$\begin{aligned}\mathcal{L}(h) &= \sum (-1)^i \,\mathrm{Trace}(h \mid H^i_c(X^h)) \\ &= \sum (-1)^i \dim H^i_c(X^h)\end{aligned}$$

For the general case we decompose h and use induction $\square$