Recall that \( X(e) = G^F / B^F \) so that \( R_e (1_F) = \text{Ind}_{B^F}^{G^F} (1_B) \)

\[ \sum_{x \in \text{Irr} W^F} x(1) \rho_x \]

where the \( \rho_x \)'s are the principal series unipotent characters.

1) **Unipotent characters**

- **Def:** An irreducible character of \( G^F \) is unipotent if it is a constituent of \( R_w = \sum (-1)^i H^i_c (X(w)) \)
  
  for some \( w \in W \). We denote by \( \text{Uch}(G^F) \) the set of unipotent characters.

Note that \( R_w \) involves in general many characters,

since \( \langle R_w, R_w \rangle_{G^F} = |C_W(wF)| \)

**Aim:** Define linear combinations of \( R_w \)'s which are closer to being irreducible characters.

ex: \( 1_{G^F} = \frac{1}{|W|} \sum R_w \)
2) **Almost characters**

From now on, assume that Facts trivially on $W$

def: given $X \in \text{Ir} W$ the almost character $R_X$ is

$$R_X = \frac{1}{|W|} \sum_{w \in W} X(w) R_w$$

(by definition it is a uniform function)

$\text{Ex.}$: $R_{1_w} = 1_{GF}$ and $R_{\text{sgn}_W} = St_{GF}$ Steinberg char.

$\text{Prop.}$: $\langle R_X; R_{X'} \rangle_{GF} = \delta_{X, X'}$

$\text{proof:}$

$$\langle R_X; R_{X'} \rangle = \frac{1}{|W|^2} \sum_{w, w'} X(w) \overline{X'(w')} \langle R_w; R_{w'} \rangle$$

$$= \frac{1}{|W|^2} \sum_{w, w'} X(w) \overline{X'(w')} |C_W(w)|$$

$$= \frac{1}{|W|} \sum_{w \in W} X(w) \overline{X'(w)} = \langle \chi; \chi' \rangle_W \Box$$

As a consequence we get an inversion formula $R_w = \sum_{X \in \text{Ir} W} X(w) R_X$

$\leftrightarrow$ the study of $R_w$'s $\leftrightarrow$ the study of almost characters.
3) **Examples**

- \( G = \text{SL}_2, \text{GL}_2 \)
  \[
  \begin{align*}
  R_e &= 1 + \text{St} \\
  R_s &= 1 - \text{St}
  \end{align*}
  \]
  \( \iff \)
  \[
  \begin{align*}
  R_1 &= \frac{1}{2}(R_e + R_s) = 1 \\
  R_{\text{sgn}} &= \frac{1}{2}(R_e - R_s) = \text{St}
  \end{align*}
  \]

- \( G = \text{GL}_3 \)
  \( \text{Irr} W = \{ 1_w = \chi(3); \chi(21); \chi(1^3) \} \)
  \[
  \begin{align*}
  R_e &= \chi(3) + 2\chi(21) + \chi(1^3) \\
  R_s &= \text{Harish-Chandra induction of } \frac{\text{R}_{\text{GL}_2 \times \text{G}_m}}{}
  \end{align*}
  \]
  \[
  \Rightarrow R_s = (\chi(3) + \chi(21)) - (\chi(1^1) + \chi(1^2)) \quad \text{or } \quad \chi(2) - \chi(1^2)
  \]
  \[
  = \chi(3) - \chi(1^2)
  \]
  \[
  \Rightarrow R_{\text{st}} = a\chi(3) + b\chi(21) + c\chi(1^2) + \chi \text{ with } \chi \text{ not in the principal series of } \{ \chi(3); \chi(21); \chi(1^3) \}
  \]

We use the orthogonality relations:

- \( \langle R_e; R_{\text{st}} \rangle = 0 \Rightarrow a + 2b + c = 0 \)
- \( \langle R_s; R_{\text{st}} \rangle = 0 \Rightarrow a - c = 0 \)
- \( \langle R_{\text{st}}; R_{\text{st}} \rangle = 3 \Rightarrow a^2 + b^2 + c^2 + \langle \rho, \rho \rangle = 3 \)
\[ \Rightarrow a = -b = c = 1 \text{ and } \langle \rho; \rho \rangle = 0 \]


\[ \text{i.e. } R_{st} = \rho(2) - \rho(11) + \rho(1^3) \]

We deduce the almost characters:

\[ R_{X(3)} = \rho(3) \land R_{X(21)} = \rho(21) \land R_{X(1^3)} = \rho(1^3) \]

**Thm:** For \( G = \text{GL}_n \) (or any reductive group of type A)

- every unipotent character lies in the principal series
- and \( \forall X \in \text{Irr}W \quad R_X = \rho_X \)

4) **Families of characters**

Kazhdan-Lusztig defined:

- a partition of \( W \) into 2-sided cells + order
- a partition of \( \text{Irr}W \) into families + order
- an order preserving bijection from two-sided cells to families

**def:** a family of unipotent characters of \( G^F \) is the

set of irreducible constituents of \( R_X \) where \( X \) runs in a given family of \( \text{Irr}W \)

**Ex:** Families of \( \mathbb{Z}_n \) are singletons

\[ \text{unipotent char. of } \text{GL}_n(q) \]

\[ \rho_X \]

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Finally, to each family $\mathcal{F}$ is attached a "small" finite gp $A_F$

Let $M(A_F) = \{ (a, \psi) \mid a \in A_F, \psi \in \text{Irr} A_F \}$

and $\{ (a, \psi), (b, \psi') \} = \sum_{\substack{x \in A_F \setminus C_{A_F}(a) \setminus C_{A_F}(b) \cap C_{A_F}(a) \setminus C_{A_F}(b) \cap C_{A_F}(a)}} \frac{\Psi(x') \overline{\psi(x)}}{|C_{A_F}(a)||C_{A_F}(b)|} \psi(a)$

Lusztig's classification thm: There is

* a bijection $M(A_F) \leftrightarrow \text{Uch}\mathfrak{F}$

$(a, \psi) \mapsto \rho((a, \psi))$

* an embedding $\mathfrak{F} \hookrightarrow M(A_F)$ s.t.

$\langle \rho((a, \psi)), R_\chi \rangle_{G_F} = \{ (a, \psi), \chi(x) \}$

The matrix $\{ (a, \psi), (b, \psi') \} \ (a, \psi), (b, \psi') \in M(A_F)$

is called the Fourier matrix attached to $\mathcal{F}$.

For some families in $E_7$ and $E_8$, a sign needs to be added

Particular case: if $A_F$ is abelian, $M(A_F) \cong A_F \times \text{Irr} A_F$

and $\{ (a, \psi), (b, \psi') \} = \frac{1}{|A_F|} \psi(b) \overline{\chi(a)}$

is the usual Fourier transform