IV-2 UNIPOTENT CHARACTERS

Recall that $X(e) = G^F/B^F$ so that $R_e(1_{T^F}) = \mathrm{Ind}_{B^F}^{G^F}(1_{B^F})$

I-4 $\Rightarrow$ decomposition of $R_e = \sum_{\chi \in \mathrm{Irr} W^F} \chi(1)\, \rho_\chi$

where the ρ_χ's are the principal series unipotent characters

1) Unipotent characters

def: an irreducible character of G^F is unipotent if it is a constituent of $R_w = \sum (-1)^i H^i_c(X(w))$ for some $w \in W$. We denote by $\mathrm{Uch}(G^F)$ the set of unip chars.

Note that R_w involves in general many characters since $\langle R_w ; R_w \rangle_{G^F} = |C_W(wF)|$

Aim: define linear combinations of R_w's which are closer to being irreducible characters

ex: $1_{G^F} = \frac{1}{|W|} \sum_w R_w$

2) Almost characters

From now on, assume that F acts trivially on W

def: given $\chi \in \mathrm{Irr}\, W$ the almost character R_χ is

$$R_\chi = \frac{1}{|W|} \sum_{w \in W} \chi(w) R_w$$

(by definition it is a uniform function)

Ex: $R_{1_W} = 1_{G^F}$ and $R_{\mathrm{sgn}_W} = \mathrm{St}_{G^F}$ Steinberg char.

Prop: $\langle R_\chi ; R_{\chi'} \rangle_{G^F} = \delta_{\chi,\chi'}$

proof: $\langle R_\chi ; R_{\chi'} \rangle = \frac{1}{|W|^2} \sum_{w,w'} \chi(w) \overline{\chi'(w')} \langle R_w ; R_{w'} \rangle$

$$= \frac{1}{|W|^2} \sum_{\substack{w,w' \\ w \sim w'}} \chi(w) \overline{\chi'(w)} \, |C_W(w)|$$

$$= \frac{1}{|W|} \sum_{w \in W} \chi(w) \overline{\chi'(w)} = \langle \chi ; \chi' \rangle_W \quad \square$$

As a consequence we get an inversion formula $R_w = \sum_{\chi \in \mathrm{Irr} W} \chi(w) R_\chi$

$\rightsquigarrow$ the study of R_w's $\Longleftrightarrow$ the study of almost characters.

3) Examples

- $G = Sl_2, Gl_2$

$$\left.\begin{array}{l} R_e = 1 + St \\ R_s = 1 - St \end{array}\right) \iff \left(\begin{array}{l} R_1 = \frac{1}{2}(R_e + R_s) = 1 \\ R_{sgn} = \frac{1}{2}(R_e - R_s) = St \end{array}\right.$$

- $G = Gl_3 \quad \mathrm{Irr}\, W = \{ 1_W = \chi_{(3)} ; \chi_{(21)} ; sgn_W = \chi_{(1^3)} \}$

$$R_e = \rho_{(3)} + 2\rho_{(21)} + \rho_{(1^3)}$$

$s = (1,2) \in \mathfrak{S}_2$ parabolic subgroup of $\mathfrak{S}_3$

$\Rightarrow R_s =$ Harish-Chandra induction of $\underbrace{R_s^{Gl_2 \times \mathbb{G}_m}}_{\rho_{(2)} - \rho_{(1^2)}}$

$$\Rightarrow R_s = \left(\rho_{(3)} + \rho_{(21)}\right) - \left(\rho_{(21)} + \rho_{(1^3)}\right)$$
$$= \rho_{(3)} - \rho_{(1^3)}$$

$\underbrace{R_{st}}_{(1,2,3)} = a\rho_{(3)} + b\rho_{(21)} + c\rho_{(1^3)} + \rho$ with ρ not in the principal series $\{\rho_{(3)} ; \rho_{(21)} ; \rho_{(1^3)}\}$

We use the orthogonality relations:

$$\langle R_e ; R_{st} \rangle = 0 \quad \Rightarrow \quad a + 2b + c = 0$$
$$\langle R_s ; R_{st} \rangle = 0 \quad \Rightarrow \quad a - c = 0$$
$$\langle R_{st} ; R_{st} \rangle = 3 \quad \Rightarrow \quad a^2 + b^2 + c^2 + \langle \rho, \rho \rangle = 3$$

$\Rightarrow a=-b=c=1$ and $\langle \rho;\rho\rangle = 0$

i.e. $R_{st} = \rho_{(3)} - \rho_{(21)} + \rho_{(1^3)}$

We deduce the almost characters:

$$R_{\chi_{(3)}} = \rho_{(3)} \; ; \; R_{\chi_{(21)}} = \rho_{(21)} \; ; \; R_{\chi_{(1^3)}} = \rho_{(1^3)}$$

<u>Thm</u>: For $G = GL_n$ (or any reductive group of type A)

every unipotent char. lies in the principal series and $\forall \chi \in \mathrm{Irr}W$ $R_\chi = \rho_\chi$

4) Families of characters

Kazhdan-Lusztig defined:

- a partition of W into 2 sided cells + order
- a partition of $\mathrm{Irr}W$ into families + order
- an order preserving bijection from two-sided cells to families

def: a family of unipotent characters of G^F is the set of irreducible constituents of R_χ where χ runs in a given family of $\mathrm{Irr}W$

<u>Ex</u>: Families of $\mathfrak{S}_n$ are singletons $\qquad \chi_\lambda \updownarrow \rho_\lambda$

——— unipotent char. of $GL_n(q)$ ———

Finally, to each family $\mathcal{F}$ is attached a "small" finite gp $A_\mathcal{F}$

Let $\mathcal{M}(A_\mathcal{F}) = \{(a,\psi) \mid a \in A_\mathcal{F}, \psi \in \operatorname{Irr} A_\mathcal{F}\}/\sim$

and
$$\{(a,\psi);(b,\varphi)\} := \sum_{\substack{x \in A_\mathcal{F} \\ {}^x a \in C_{A_\mathcal{F}}(b)}} \frac{\psi(x^{-1} b x)\,\overline{\varphi(xax)}}{|C_{A_\mathcal{F}}(a)|\,|C_{A_\mathcal{F}}(b)|}$$

<u>Lusztig's classification thm</u>: There is

* a bijection $\mathcal{M}(A_\mathcal{F}) \longleftrightarrow \operatorname{Uch}\mathcal{F}$
  $(a,\psi) \longmapsto \rho_{(a,\psi)}$
* an embedding $\mathcal{F} \xhookrightarrow{i} \mathcal{M}(A_\mathcal{F})$ s.t

$$\langle \rho_{(a,\psi)} ; R_\chi \rangle_{G^F} = \{(a,\psi); i(\chi)\} \qquad \pm$$

The matrix $\{(a,\psi);(b,\varphi)\}_{(a,\psi),(b,\varphi) \in \mathcal{M}(A_\mathcal{F})}$ is called the Fourier matrix attached to $\mathcal{F}$.

? For some families in E_7 and E_8, a sign needs to be added

<u>Particular case</u>: if $A_\mathcal{F}$ is abelian, $\mathcal{M}(A_\mathcal{F}) \simeq A_\mathcal{F} \times \operatorname{Irr} A_\mathcal{F}$

and
$$\{(a,\psi);(b,\varphi)\} = \frac{1}{|A_\mathcal{F}|}\,\psi(b)\,\overline{\varphi(a)}$$

is the usual Fourier transform