IV-3 THE COXETER VARIETY

Here we study $H_c^\bullet(X(w))$ for w a Coxeter element
We will assume that
- F acts trivially on W
- W is irreducible

1) Coxeter elements

def: a Coxeter element of W is the product
$c = s_1 s_2 \cdots s_r$ of all the simple reflections $S = \{s_1, \ldots, s_r\}$
in any order

Ex: $c = (1,2,\ldots,n) = (1,2)(2,3)\cdots(n-1,n)$ is a Coxeter elt of $\mathfrak{S}_n$

Prop: Let w be a Coxeter elt of W
(i) $C_W(c) = \langle c \rangle$
(ii) Every Coxeter elt of W is conjugate to w
(and by a sequence of cyclic shifts)
(iii) The order h of c, called the Coxeter number
satifies $\#S \times h = 2 \times \#\mathrm{Ref}(W) = 2\,\ell(w_0)$

Rmk: a cyclic shift is $v \cdot v' \longmapsto v' \cdot v$ where $\ell(vv') = \ell(v) + \ell(v')$

Here is a list of the Coxeter numbers:

type	A_n	B_n/C_n	D_n	E_6	E_7	E_8	F_4	G_2
h	$n+1$	$2n$	$2n-2$	12	18	30	12	6

2) Coxeter varieties

def: A Coxeter variety is a Deligne Lusztig variety attached to a Coxeter element

Ex: $X((1,2,\dots,n)) = \mathbb{P}_{n-1} \setminus$ all rational hyperplanes (see II-4) is a Coxeter variety

The following geometric properties will be used to compute the cohomology of any Coxeter variety

Prop: Let c be a Coxeter element

(i) $X(c)$ is irreducible and affine

(ii) $\forall\, i = 1, \dots, h-1 \quad X(c)^{F^i} = \emptyset$ and $\# X(c)^{F^h} = |G^F| / |T^{cF}|$

(iii) Let $I \subseteq S$ and $c_I = \prod_{s \in I} s$ (Coxeter elt of W_I)

Then

$$\underset{\substack{\uparrow \\ \text{unipotent radical of } P_I}}{U_I^F} \backslash X(c) \simeq \underbrace{X_{L_I}(c_I)}_{\text{Coxeter variety of } L_I} \times (\mathbb{G}_m)^{|S \setminus I|}$$

3) Cohomology of $X(c)$ - Case of GL_n

Here $c=(1,2,\dots,n)$

Recall that for $GL_n(q)$ the unipotent characters coincide with the almost characters so that

$$R_c = \sum (-1)^i H^i_c(X(c)) = \sum_{\lambda \vdash n} \chi_\lambda(c)\, \underbrace{\rho_\lambda}_{R_{\chi_\lambda}}$$

Now $\chi_\lambda(c) \neq 0 \Rightarrow \lambda = (i\, 1^{n-i})$ in which case

$$\chi_\lambda(c) = (-1)^{n-i}$$

$$\rightsquigarrow R_c = \underbrace{\rho_{(n)}}_{1_{GL_n(q)}} - \rho_{(n-1,1)} + \rho_{(n-2,1^2)} + \cdots + (-1)^{n-1} \underbrace{\rho_{(1^n)}}_{St_{GL_n(q)}}$$

<u>Thm</u>: The cohomology of $X_n = X((1,\dots,n))$ is given by

i	$n-1$	$n-2$	$n-3$	$\cdots\cdots$	$2n-3$	$2n-2$
H^i_c	$\rho_{(1^n)}$	$\rho_{(2,1^{n-2})}$	$\rho_{(3,1^{n-3})}$		$\rho_{(n-1,1)}$	$\rho_{(n)}$
F	1	q	q^2	$\cdots\cdots$	q^{n-2}	q^{n-1}

<u>proof</u>: By induction on n

<u>$n=2$</u>: $X_2 = \mathbb{P}_1 \setminus \mathbb{P}_1(\mathbb{F}_q)$ and we already computed the cohomology: $H^\bullet_c(X_2) = St[-1] \oplus 1[-2]$

with eigenvalues of F equal to 1 and q

Assume the thm holds for X_{n-1} for some n

Let $I = \{(1,2), \dots, (n-2,n-1)\} \subseteq S$ so that $L_I = GL_{n-1} \times GL_1$

$$\rightsquigarrow \quad U_I^F \backslash X_n \simeq X_{n-1} \times \mathbb{G}_m$$

which induces $H_c^\bullet(X_n)^{U_I^F} \simeq H_c^\bullet(X_{n-1}) \otimes H_c^\bullet(\mathbb{G}_m)$

Harish-Chandra restriction

We get :

i	$n-1$	n	$n+1$	$\dots$
$H_c^i(X_n)^{U_I^F}$	$\rho_{(1^{n-1})} + \rho_{(2,1^{n-3})}$	$\rho_{(2,1^{n-2})} + \rho_{(3,1^{n-3})}$	$\rho_{(3,1^{n-3})} + \rho_{(4,1^{n-4})}$	$\dots$
F	$1, 1$	q, q	q^2, q^2	$\dots$

Harish-Chandra restriction coincides with restriction on almost characters and we conclude using the fact that

$$\mathrm{Res}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n} \chi = \chi_{(k+1,1^{n-k-2})} + \chi_{(k,1^{n-k-1})} \iff \chi = \chi_{(k+1,1^{n-k-1})}$$

(the case $n=3$ has to be treated separately using the value of $R_c = \rho_{(3)} - \rho_{(2,1)} + \rho_{(1^3)}$) $\square$

4) Cohomology of $X(c)$ – General case

Recall that $\# X(c)^{F^k} = \sum (-1)^i \operatorname{Tr}(F^k \mid H^i_c(X(c))) = 0$
for all $k = 1, \dots, h-1$

$\Rightarrow$ F has at least h different eigenvalues on $H^\bullet_c(X(c))$

<u>Thm</u>: F has exactly h eigenvalues on $H^\bullet_c(X(c))$
The corresponding (generalized) eigenspaces are mutually non-isomorphic irreducible representations of G^F

$\Rightarrow$ These are the representations in $R_c = \sum (-1)^i H^i_c(X(c))$
$\langle R_c ; R_c \rangle = \# C_W(c) = h$ is upgraded to
$$\operatorname{End}_{G^F}(H^\bullet_c(X(c))) \simeq \overline{\mathbb{Q}}_\ell[t] / \underbrace{(t-\lambda_1)\cdots(t-\lambda_r)}_{\text{eigenvalues of } F}$$

Important ingredient in the proof: if ρ is cuspidal
(does not occur in $H^\bullet_c(X(w))$ for $w \in W_I$ $I \subsetneq S$)
then $\langle H^i_c(X(c)) ; \rho \rangle = 0$ if $i \neq \ell(c)$

<u>proof</u>: $\overline{X(c)} = \bigsqcup_{v \leq c} X(v)$ is smooth

Moreover $v < c \Rightarrow v$ lies in a proper parabolic subgp of W

Therefore $X(c) \hookrightarrow \overline{X}(c)$ gives

$$\langle H^i_c(X(c)); \rho \rangle = \langle H^i_c(\overline{X(c)}); \rho \rangle$$
$$= \langle H_c^{2\ell(c)-i}(\overline{X(c)}); \rho^* \rangle \quad (\text{Poincaré})$$
$$= \langle H_c^{2\ell(c)-i}(X(c)); \rho \rangle$$

But $X(c)$ is affine $\Rightarrow$ $H_c^{2\ell(c)-i}(X(c)) = 0$ if $i > \ell(c)$
$H^i_c(X(c)) = 0$ if $i < \ell(c)$