IV-4 CONJECTURES ON THE COHOMOLOGY OF DL VARIETIES

Assume again for simplicity that $W^F = W$

1) Motivation : the principal series

Recall from day 1 that we decomposed $\mathrm{Ind}_{B^F}^{G^F} 1_{B^F}$ using a natural isomorphism

$$\mathrm{End}_{G^F}(\mathbb{C}G^F/B^F) \simeq \mathcal{H}_q(W) \quad \overset{q=1}{\rightsquigarrow} \quad \mathbb{C}W$$

inducing

$$\begin{array}{ccc} \mathrm{Irr}\,\mathbb{C}G^F/B^F & \longleftrightarrow & \mathrm{Irr}\,W \\ \rho_\chi & \longleftarrow & \chi \end{array}$$

such that

$$[\mathbb{C}G^F/B^F] = \sum_{\chi \in \mathrm{Irr} W} \chi(1)\, \rho_\chi$$

Now $\mathbb{C} \leftrightarrow \overline{\mathbb{Q}}_\ell$ and $\overline{\mathbb{Q}}_\ell G^F/B^F = H_c^\bullet(X(e))$

$\rightsquigarrow$ decomposition of the DL character

$$R_e = \sum_{\chi \in \mathrm{Irr} W} \chi(1)\, \rho_\chi$$

<u>Generalisation</u>? Since $\langle R_w ; R_w \rangle_{G^F} = |C_W(w)|$

one could hope for a natural isomorphism

$$\mathrm{End}_{G^F}\left(H_c^\bullet(X(w))\right) \simeq \mathcal{H}\left(C_W(w)\right)$$

some version of a Hecke algebra

inducing $\mathrm{Irr}\, H_c^\bullet(X(w)) \longleftrightarrow \mathrm{Irr}\, C_W(w)$

$$\rho_\chi \longleftarrow \chi$$

such that

$$R_w = \sum \pm \chi(1)\, \rho_\chi$$

sign depending on the degree of $H_c^\bullet$ in which ρ_χ occurs

<u>Example</u>: $w = (1, 2, \dots, n)$ Coxeter el^t of $\mathfrak{S}_n$

$$\mathrm{End}_{G^F}\left(H_c^\bullet(X(w))\right) \simeq \overline{\mathbb{Q}}_\ell[t] \big/ (t-1)(t-q)\cdots(t-q^{n-1})$$

$$F \longleftarrow t$$

is a Hecke algebra associated to the cyclic group $C_W(w) \simeq \mathbb{Z}/n\mathbb{Z}$

$$\mathrm{Irr}\, H_c^\bullet(X(w)) \longleftrightarrow \mathrm{Irr}\, \mathbb{Z}/n\mathbb{Z}$$

$$\rho\ \ ]_{i+1} \longleftarrow \left(1 \longmapsto \exp(2ik\pi/n)\right)$$

such that $R_w = \sum_{i=0}^{n-1} (-1)^{n-1-i} \rho$

<u>Rmk</u>: $q \longmapsto \sqrt[n]{1}$ yields $\overline{\mathbb{Q}}_\ell[t] \big/ (t-1)\cdots(t-q^{n-1}) \rightsquigarrow \overline{\mathbb{Q}}_\ell\, \mathbb{Z}/n\mathbb{Z}$

2) Conjectures

R_w depends only on the conjugacy class of w but not $H_c^\bullet(X(w))$!

Ex : $G = GL_3$

- $w = (1,2)$ $X(w)$ is a curve "induced" from the Coxeter variety of $GL_2 \times GL_1$

- $w = (1,3) = (2,3)(1,2)(2,3)$ $X(w)$ has dimension 3

i	1	2	3	4	5	6
$H_c^i(X(1,2))$	$\rho_{(1,1,1)} + \rho_{(2,1)}$	$\rho_{(2,1)} + \rho_{(3)}$	·	·	·	·
$H_c^i(X(1,3))$	·	·	$\rho_{(1,1,1)}$	·	·	$\rho_{(3)}$

$C_{\mathfrak{S}_3}((1,2)) \simeq C_{\mathfrak{S}_3}((1,3)) \simeq \mathbb{Z}/2\mathbb{Z}$ $\quad \mathrm{End}_{G^F}(H_c^\bullet(X(1,2)))$ too big!

$\qquad$ ——— $(1,3)$ OKAY

def: We set $\pi = w_0 \cdot w_0 \in B_W^+$

product in the Braid monoid

Rmk : π is central in B_W^+

The elements we should look at are roots of π

Ex: $W = \mathfrak{S}_2$ $\quad \pi = (s_1 s_2 s_1) \cdot (s_1 s_2 s_1)$

$$\Rightarrow (s_1 s_2)^3 = s_1 s_2 s_1 \underbrace{s_2 s_1 s_2}_{s_1 s_2 s_1} = \pi$$

More generally, if $c \in W$ is a Coxeter element then $c^h = \pi$

Conjecture [Broué-Michel] Let $w \in B_W^+$ s.t $w^d = \pi^a$ $d, a \geq 1$

(i) $\langle H_c^i(X(w)), H_c^j(X(w)) \rangle_{G^F} = 0$ if $i \neq j$

(ii) The action of $C_{B_W^+}(w)$ on $X(w)$ induces a surjective map $\overline{\mathbb{Q}}_\ell C_{B_W}(w) \twoheadrightarrow \mathrm{End}_{G^F}(H_c^\bullet(X(w)))$

(iii) $C_W(w)$ is a complex reflection group and $C_{B_W}(w) \simeq B_{C_W(w)}$

(iv) The map in (ii) factors through a Hecke algebra

$$\mathcal{H}(C_W(w)) \xrightarrow{\sim} \mathrm{End}_{G^F}(H_c^\bullet(X(w)))$$

s.t $\mathcal{H}(C_W(w))|_{q = \sqrt[h]{1}} = \overline{\mathbb{Q}}_\ell C_W(w)$

Rmk: (iii) was recently proven by Digne-Michel

The specialization $q = \sqrt[h]{1}$ will become natural when working over $\overline{\mathbb{F}}_\ell$ instead of $\overline{\mathbb{Q}}_\ell$

In addition, there is a conjectural explicit description of $H_c^\bullet(X(w))$ (as a graded G^F-module)

4) <u>The case of π^a</u>

In the usual Hecke algebra $\mathcal{H}_q(W) = \langle h_w \mid \cdots \rangle$ the elt $h_\pi = h_{w_0} \cdot h_{w_0}$ acts by scalars on the Kazhdan-Lusztig basis

<u>Thm</u> (Bonnafé-D.-Rouquier) $\forall w \in W$

$$\langle H_c^i(X(\pi^a w)) ; \rho \rangle = \langle H_c^{i-2a \cdot n_\rho}(X(w)), \rho \rangle$$

with n_ρ being an explicit constant attached to ρ

$\rightsquigarrow$ $H_c^\bullet(X(\pi))$ is just a shifted version of $H_c^\bullet(X(1)) = \overline{\mathbb{Q}}_\ell G^F/B^F$