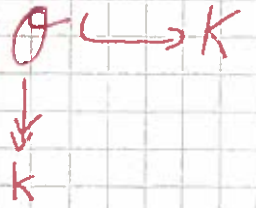


V-1 Blocks and defect groups

Fix ~~a~~ ^{finite} group G and a prime number l

Def: An l -modular system is (K, \mathcal{O}, k) where

- \mathcal{O} is a complete d.v.r with max. ideal \mathfrak{m}
- $K = \text{Frac}(\mathcal{O})$
- $k = \mathcal{O}/\mathfrak{m}$ field of char. l



Ex: ~~($\mathbb{R}, \mathbb{Z}, \mathbb{F}_2$)~~ $(\mathbb{Q}_l, \mathbb{Z}_l, \mathbb{F}_l)$

Def: ~~Answer~~ We say that (K, \mathcal{O}, k) is big enough if KG and kG split.

Recall that KG is split if all simple modules remain simple after extending the field.

Rem: • let $K = \mathbb{Q}_l(\zeta)$, ζ prim. l th root of unity.
 \mathcal{O} its ring of integers with residue field k .
Then (K, \mathcal{O}, k) is big enough for G .

Comparing KG -mod and kG -mod

finite dim. representations

- * KG -mod is semisimple: it's enough to understand the simple objects $\text{Irr } KG$. They are det. by their character
 - * kG -mod is semisimple if and only if $l \nmid |G|$.
- \Rightarrow if $l \mid |G|$ many interesting classes of objects: simple, indec., proj. modules + homological information.

Ex: let $G = C_l = \langle g \rangle$, $g^l = 1$ and $M \in kG$ indec.

Then $g \in \text{End}_k(M)$ sat. $g^l m = m$ for all $m \in M$.

Therefore, $(g^l - 1)m = (g - 1)^l m = 0$.

So $g - 1$ is nilpotent, up to isomorphism $M = k^{\oplus r}$, $1 \leq r \leq l$

and $g \cdot m = \text{Fr}(1) \cdot m$ for all $m \in M$,

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where $J_r(1) = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$.

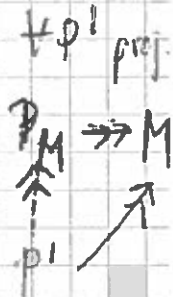
Note that M is simple iff $r=1$ ($\sim J_r(kQ) = \{k\}$)
 and M is projective iff $r=l$. (In this case $M \cong kP$).

Lifting projective modules

Fact: Every kG -module M has a proj. cover $P_M \rightarrow M$.

In addition, P_M lifts to a proj. σG -mod \tilde{P}_M s.t. \exists

$$k\tilde{P}_M := k \otimes_{\sigma} \tilde{P}_M \cong P_M. \text{ (lift unique up to isomorphism.)}$$



Prop: Let $P, Q \in kG$ -mod projective. Then

$$P \cong Q \iff k\tilde{P} \cong k\tilde{Q}$$

\Rightarrow The character of $k\tilde{P}$ determines P .

2) Blocks and defect groups

Def: A block of kG (or σG) is an indec. direct summand of kG or σG as a (G, G) -bimodule.

Fact: $\sigma G \rightarrow kG$ induces a bijection of blocks.

Ex: $G = S_3$ $l=2$: $kG \cong B \oplus \text{Mat}_2(k)$

$l=3$: kG is indec.

$l=3$: $kG \cong k \oplus k \oplus \text{Mat}_2(k)$

If $\sigma G \cong \bigoplus_i B_i$ then $\sigma G \cong \prod_i B_i$ as σ -algebras $\left[\begin{array}{l} \text{simple or idemp.} \\ \text{so} \end{array} \right]$

$$\sigma G\text{-mod} \cong \bigoplus_i B_i\text{-mod.}$$

This decomp. (tens. by k or K) induces partitions

$$\text{Irr } kG = \bigsqcup \text{Irr } kB_i$$

$$\text{Irr } KG = \bigsqcup \text{Irr } KB_i$$

We say: $x \in \text{Irr } KG$ belongs to B if $x \in \text{Irr } KB$

Def.: The principal block B_0 is the block to which 1_G belongs, ~~the~~.

Ex.: $G = S_3$ $l=2$: $\text{Irr } KG = \{ \chi_{\square\square}, \chi_{\square} \} \sqcup \{ \chi_{\square\square} \}$

$l=3$: only one block

$l > 3$: $\text{Irr } KG = \{ \chi_{\square\square} \} \sqcup \{ \chi_{\square} \} \sqcup \{ \chi_{\square\square} \}$

To each block B one can attach an l -subgroup D of G called a defect group of B .

more precisely:
the G -conj.
class of an
 l -group

We have

$$|D| = \max \left\{ \left(\frac{|G|}{\pi(l)} \right)_l \mid \chi \in \text{Irr}(KB) \right\}$$

assuming split

Example. • ~~Defect group~~ $OB \cong \text{Mat}_n(\mathbb{F})$ for some $n \iff D=1$

• defect groups of principal block are always Sylow subgroups

3) Principal blocks for finite red groups

[BHM]

Setting. G conn. red. group / $\overline{\mathbb{F}}_p$, $F: G \rightarrow G$ Frobenius.

T quasi-split max. torus of G , $W = N_G(T)H$
(contained in F -stable torus)

Fact. If $l \neq p$ and $l > l_i$ (= Cox. number = order of Cox. element)
then Sylow l -subgroups of GF are abelian.

\implies Sylow l -subgroups are contained in tori.

Thm. let D be a Sylow l -subgroup of GF and $w \in W$
s.t.h. $D \subset TW^F$. Assume $C_G(D)$ max. torus.

Then the unipotent characters in the principal block are the ^{irred.} constituents of $R_w (= R_w(1))$.

" p -Harish-Chandra theory"

(3)

Example: $G = GL_n$ and $l > n$ (= Coxeter number)

- If $l \mid q-1 (= \Phi_1(q))$ then $TF \cong l$ -Sylow and the principal block contains all the unipotent characters
- If $l \nmid \Phi_1(q)$ then $T^{(1,2,\dots,n)}F \cong TF_q^X \cong l$ -Sylow and the unipotent chars in the princ. block are

In GL_n :
unipotent characters = principal unipotent characters

$$1_G F = e_{\square}, e_{\square}, e_{\square}, \dots, e_{\square} = St_G F$$

[Recall from Oliver's talk:

$$w = (1, 2, \dots, n) \quad R_w = \sum_{\lambda \vdash n} \kappa_\lambda(w) e_\lambda \quad \kappa_\lambda(w) \neq 0 \text{ iff } \lambda = (i, 1^{n-i})]$$