V - 1 Blocks and defect groups

Fix a finite group G and a prime number ℓ.

Def: An ℓ-modular system is $(K, \mathcal{O}, k)$ where

- $\mathcal{O}$ is a complete d.v.r with max. ideal m
- $K = \mathrm{Frac}(\mathcal{O})$
- $k = \mathcal{O}/m$ field of char. ℓ

$$\begin{tikzcd} \mathcal{O} \arrow[r, hook] \arrow[d, two heads] & K \\ k & \end{tikzcd}$$

Ex: $(\mathbb{Q}_\ell, \mathbb{Z}_\ell, \mathbb{F}_\ell)$

Def: We say that $(K, \mathcal{O}, k)$ is big enough if KG and kG split.

Recall that KG is split if all simple modules remain simple after extending the field.

Rem: • Let $K = \mathbb{Q}_\ell(\zeta)$, ζ prim. $|G|$th root of unity, $\mathcal{O}$ its ring of integers with residue field k. Then $(K, \mathcal{O}, k)$ is big enough for G.

Comparing KG-mod and kG-mod

finite dim. representations

* KG-mod is semisimple: it's enough to understand the simple objects in KG. They are det. by their character — Maschke

* kG-mod is semisimple if and only if $\ell \nmid |G|$
$\rightsquigarrow$ if $\ell \mid |G|$ many interesting classes of objects: simple, indec., proj. modules + homological information

Ex. Let $G = C_\ell = \langle g \rangle$, $g^\ell = 1$ and $M \in kG$ indec.
Then $g \in \mathrm{End}_k(M)$ sat. $g^\ell m = m$ for all $m \in M$.
Therefore, $(g^\ell - 1)m = (g-1)^\ell m = 0$.
So $g-1$ is nilpotent. Up to isomorphism $M = k^r$, $1 \leq r \leq \ell$
and $g \cdot m = J_r(1) \cdot m$ for all $m \in M$,

where $J_r(1) = \begin{pmatrix} 1 & 1 & & \\ & \ddots & \ddots & \\ & & & 1 \\ & & & 1 \end{pmatrix}$.

Note that M is simple iff $r=1$ ($\rightsquigarrow \operatorname{Irr}(kC_\ell)=\{k\}$) and M is projective iff $r=\ell$. (In this case $M \simeq kP$).

Lifting projective modules

Fact: Every kG-module M has a proj. cover $P_M \twoheadrightarrow M$. In addition, P_M lifts to a proj. $\mathcal{O}G$-mod $\tilde{P}_M$ s.th. $\exists$ $k\tilde{P}_M := k \otimes_{\mathcal{O}} \tilde{P}_M \simeq P_M$. (lift unique up to isomorph.)

$\forall P'$ proj

$$\begin{tikzcd} P_M \arrow[r, two heads] & M \\ P' \arrow[u, dashed] \arrow[ur] & \end{tikzcd}$$

Prop: Let $P, Q \in kG$-mod projective. Then
$$P \cong Q \iff K\tilde{P} \simeq K\tilde{Q}$$
$\Rightarrow$ The character of $K\tilde{P}$ determines P.

2) Blocks and defect groups

Def: A block of kG (or $\mathcal{O}G$) is an indec. direct summand of kG or $\mathcal{O}G$ as a (G,G)-bimodule.

Fact: $\mathcal{O}G \twoheadrightarrow kG$ induces a bijection of blocks.

Ex.: $G = S_3$
- $\ell=2$: $kG \simeq B \oplus \mathrm{Mat}_2(k)$
- $\ell=3$: kG is indec.
- $\ell>3$: $kG \simeq k \oplus k \oplus \mathrm{Mat}_2(k)$

If $\mathcal{O}G \cong \bigoplus_i B_i$ then $\mathcal{O}G \cong \times_i B_i$ ~~[illegible]~~ as $\mathcal{O}$-algebras ~~and~~ so ~~Therefore~~
$$\mathcal{O}G\text{-mod} \cong \bigoplus_i B_i\text{-mod}.$$

This decomp. (tens. by k or K) induces partitions
$$\operatorname{Irr} KG = \coprod \operatorname{Irr} KB_i$$
$$\operatorname{Irr} kG = \coprod \operatorname{Irr} kB_i.$$

We say: $\chi \in \mathrm{Irr}\, KG$ **belongs to** B if $\chi \in \mathrm{Irr}\, KB$

Def: The **principal block** B_0 is the block to which 1_G belongs.

Ex.: $G = S_3$

$\ell = 2$: $\mathrm{Irr}\, KG = \{\chi_{(3)}, \chi_{(1,1,1)}\} \sqcup \{\chi_{(2,1)}\}$

$\ell = 3$: only one block

$\ell > 3$: $\mathrm{Irr}\, KG = \{\chi_{(3)}\} \sqcup \{\chi_{(1,1,1)}\} \sqcup \{\chi_{(2,1)}\}$

To each block B one can attach an ℓ-subgroup D of G called **a defect group** of B. more precisely the G-conj. class of an ℓ-group

We have
$$|D| = \max\left\{ \left(\frac{|G|}{\chi(1)}\right)_\ell \;\middle|\; \chi \in \mathrm{Irr}(KB) \right\}$$

Example.
- $\mathcal{O}B \cong \mathrm{Mat}_n(\mathcal{O})$ for some $n \Leftrightarrow D = 1$ (assuming split)
- defect groups of principal block are always Sylow subgroups

3) Principal blocks for finite red. groups [BMM]

Setting: G conn. red. group $/ \overline{\mathbb{F}}_p$, $F : G \to G$ Frobenius.
T quasi-split max. torus of G (contained in F-stable Borel), $W = N_G(T)/T$

Fact: If $\ell \neq p$ and $\ell > h$ (= Cox. number = order of Cox. element) then Sylow ℓ-subgroups of G^F are abelian.
$\Rightarrow$ Sylow ℓ-subgroups are contained in tori.

Thm. Let D be a Sylow ℓ-subgroup of G^F and $w \in W$ s.th. $D \hookrightarrow T^{wF}$. Assume $C_G(D)$ max. torus. Then the unipotent characters in the principal block are the irred. constituents of R_w $(= R_w(1))$. "d-Harish-Chandra theory"

In GL_n: unipotent characters = principal unipotent characters

Example: $G = GL_n$ and $\ell > n$ (= Coxeter number)

- If $\ell \mid q-1 \,(= \Phi_1(q))$ then $T^F \supseteq \ell$-Sylow and the principal block contains all the unipotent characters
- If $\ell \mid \Phi_n(q)$ then $T^{(1,2,\dots,n)F} \simeq \mathbb{F}_{q^n}^\times \supseteq \ell$-Sylow and the unipotent chars in the princ. block are

$$1_{G^F} = e \,,\; e \,,\; e \,,\; \dots ,\; e = St_{G^F}$$

[Recall from Olivier's talk:

$$w = (1,2,\dots,n) \qquad R_w = \sum_{\lambda \vdash n} x_\lambda(w)\, e_\lambda \qquad x_\lambda(w) \neq 0 \text{ iff } \lambda = (i, 1^{n-i}) \,.]$$