V - 1 Blocks and defect groups

Fix a group $G$ and a prime number $l$

Def: an $l$-modular system is $(K, O, k)$ where
- $O$ is a complete DVR with max. ideal $m$
- $K = \text{Frac}(O)$
- $k = O/m$ field of char. $l$

Ex: $(\mathbb{F}_l, \mathbb{Z}_l, \mathbb{F}_l)$

Def: We say that $(K, O, k)$ is big enough if $K_G$ and $k_G$ split.
Recall that $K_G$ split if all simple modules remain simple after extending the field.

Rem: let $K = \mathbb{Q}(\zeta)$, $\zeta$ prim. $l$th root of unity.
$O$ its ring of integers with residue field $k$.
Then $(K, O, k)$ is big enough for $G$.

Comparing $K_G$-mod and $k_G$-mod

- $K_G$-mod is semisimple: it's enough to understand the simple objects $\text{Irr} K_G$. They are det. by their character
- $k_G$-mod is semisimple if and only if $l | |G|$. 
  => if $l | |G|$ many interesting classes of objects:
  simple, indec., proj. modules + homological information

Ex: let $G = C_2 = \langle g \rangle$, $g^2 = 1$ and $K_G$ $k_G$ indec.
Then $g \in \text{End}_k(M)$ sat. $g^l m = m$ for all $m$.
Therefore, $(g^l-1)m = (g-1)^l m = 0$.
So $g-1$ is nilpotent. Up to isomorphism $M = k[x]/(x^l)$ and $g . m = \text{tr} (1) . m$ for all $m \in M$,
where \( J_r(\lambda) = \left( \begin{array}{c} \lambda_1 \\ \vdots \\ \lambda_r \end{array} \right) \).

Note that \( M \) is simple iff \( r = d(\lambda) = \text{Ir}(kG) = \text{dim} \).
and \( M \) is projective iff \( r = l \). (In this case \( M \cong kP \).

\underline{lifting projective modules}

\textbf{Fact:} Every \( kG \)-module \( M \) has a proj. cover \( P_M \to \to M \).
In addition, \( P_M \) lifts to a proj. \( OTG \)-module \( \widetilde{P}_M \) s.th.
\[ k \widetilde{P}_M = k \otimes P_M \cong P_M. \text{ (liftings unique up to isomorphism)} \]

\textbf{Prop:} Let \( P, Q \in kG \)-mod projective. Then
\[ P \cong Q \iff K\widetilde{P} = K\widetilde{Q} \]

\( \Rightarrow \) The character of \( K\widetilde{P} \) determines \( P \).

2) Blocks and defect groups

\textbf{Def:} A \textit{block} of \( kG \) (or \( OTG \)) is an indec. direct summand of \( kG \) or \( OTG \) as a \( (G, \mathfrak{g}) \)-bimodule.

\textbf{Fact:} \( OTG \to kG \) induces a bijection of blocks.

\textbf{Ex.} \( G = S_3 \)
\( l = 2 \):
\[ kG \cong B \oplus \text{Mat}_2(k) \]
\( l = 3 \):
\[ kG \text{ is indec.} \]
\( l = 3 \):
\[ kG \cong k \oplus k \oplus \text{Mat}_2(k) \]

If \( OTG \cong \bigoplus_i B_i \) then \( OTG \cong \bigotimes_i B_i \) as \( G \)-algebras.

\textbf{Therefore:} \( OTG \)-mod \( \cong \bigoplus_i B_i \)-mod.

This decomp. (tens. by \( k \) or \( K \)) induces partitions
\[ \text{Irr } kG = \prod_i \text{Irr } KB_i; \]
\[ \text{Irr } kG = \prod_i \text{Irr } KB_i; \]
We say \( x \in \text{Im } KG \) belongs to \( B \) if \( x \in \text{Im } KB \).

**Def:** The principal block \( B_0 \) is the block to which \( 1_G \) belongs, that is:

**Ex:** \( G = S_3 \)

\( l = 2 \): \( \text{Im } KG = \left\{ x \mid x \equiv 1 \text{ mod } 2 \right\} \)

\( l = 3 \): only one block

\( l = 3 \): \( \text{Im } KG = \left\{ x \mid x \equiv 1 \text{ mod } 3 \right\} \)

To each block \( B \) one can attach an \( l \)-subgroup \( D \) of \( G \) called a defect group of \( B \).

We have

\[ |D| = \max \left\{ \left( \frac{|G|}{|N|} \right) |x \in \text{Im } (KB) \right\} \]

**Example:** \( D \cong \text{Mat}_n(\mathbb{F}) \) for some \( n \leftrightarrow D = 1 \)

- defect groups of principal block are always Sylow subgroups

3) Principal blocks for finite nd groups

Setting: \( G \) count. red. group / \( F_p \), \( F : G \rightarrow G \) Frobenius.

\( T \) quasi-split maximal torus of \( G \) \[ W = N_G(T)H \]

(continued in n-stable level)

**Fact:** If \( l \neq p \) and \( l > h \) (= ord of \( \text{Frob} \))

then Sylow \( l \)-subgroups of \( GF \) are abelian.

\( \Rightarrow \) Sylow \( l \)-subgroups are contained in tori.

**Thm:** let \( D \) be a Sylow \( l \)-subgroup of \( GF \) and we \( w \in W \)

so that \( D \rightarrow TW \cdot F \). Assume \( C_G(D) \) max. torus.

Then the unipotent characters in the principal block are the constituents of \( R_w \) (= \( R_w(1) \)).
Example: $G = GL_n$ and $k > n$ (Coxeter number).

- If $l | q - 1 = \Phi_l(q)$ then $TF$ is 2-Sylow.

And the principal block contains all the unipotent characters.

- If $l | \Phi_l(q)$ then $T_{(1^2, \ldots, 1)}(q) \cong F_{q^l}$ is the Sylow 2-group and the unipotent class in the principal block are

$$\lambda \in F = \mathbb{C} \frac{1}{1}, \mathbb{C} \frac{1}{1}, \mathbb{C} \frac{-1}{1}, \ldots, \mathbb{C} \frac{-1}{1} = \mathbb{F}$$

Recall from Olsson's talk:

$w = (1^2, \ldots, n)$

$$R_w = \sum_{\lambda \in \mathbb{F}} x_{\lambda}(w) \mathbb{C}_\lambda$$

$x_{\lambda}(w) = 0$ if

$$\lambda = (1,1^{n-1})$$