V-1 BLOCKS AND DEFECT GROUPS

1) Representations in positive characteristic

ℓ a prime number

def: a ℓ-modular system is $(K, \mathcal{O}, k)$ where
- $\mathcal{O}$ is a complete d.v.r with max. ideal $\mathfrak{m}$
- $K = \mathrm{Frac}(\mathcal{O})$ is a field of char. 0
- $k = \mathcal{O}/\mathfrak{m}$ is a field of char. ℓ

Ex: $(\mathbb{Q}_\ell, \mathbb{Z}_\ell, \mathbb{F}_\ell)$ or finite extensions

Given a finite group G, we say that the modular system $(K, \mathcal{O}, k)$ is big enough for G if KG and kG split

The categories KG-mod and kG-mod of f.d representations

* KG-mod is semisimple : it is enough to understand the simple objects $\#\mathrm{Irr}\, KG$ which are determined by their (ordinary) characters

* kG-mod : not semisimple if $\ell \mid |G|$

⇝ several interesting classes of objects : simple, indecomposable or projective representations

+ homological information : extension between objects

<u>Ex</u> : if G is an ℓ-group then $\mathrm{Irr}\, kG = \{k\}$ ↰ trivial rep.

but $\mathrm{Ext}^i_{kG}(k,k) = H^i(G,k)$

is non trivial in general !

<u>Lifting projective modules</u>

Every kG-module M has a projective cover $P_M \twoheadrightarrow M$

In addition, P_M lifts to a projective $\mathcal{O}G$-module $\tilde{P}_M$

such that $$k\tilde{P}_M := k \otimes_{\mathcal{O}} \tilde{P}_M \simeq P_M$$

The "character" determine the projective module :

<u>Prop</u> : Let P, Q be f.d projective kG-modules.

Then $$P \simeq Q \iff K\tilde{P} \simeq K\tilde{Q}$$

↑ in KG-mod

By the "character" of a projective kG-module P we will mean the character of $K\tilde{P} = K \otimes_{\mathcal{O}} \tilde{P}$

2) Blocks and defect groups

def: a block of kG or $\mathcal{O}G$ is an indecomposable direct summand of kG or $\mathcal{O}G$ as a (G,G)-bimodule

Fact: The reduction $\mathcal{O}G \twoheadrightarrow kG$ induces a bijection on blocks

Ex: $G = \mathfrak{S}_3$ $\quad \ell = 3 \quad kG$ is indecomposable
$\ell = 2 \quad kG \simeq \mathrm{Mat}_2(k) \oplus \underset{\text{dimension } 2}{B}$

If $B_1, \ldots, B_r$ are the blocks of $\mathcal{O}G$ then by definition
$$\mathcal{O}G = \bigoplus B_i \quad \text{hence} \quad \mathcal{O}G\text{-mod} = \bigoplus B_i\text{-mod}$$

This decomposition, tensored by k or K induces partitions $\mathrm{Irr}\, kG = \bigsqcup \mathrm{Irr}\, kB_i$ and $\mathrm{Irr}\, KG = \bigsqcup \mathrm{Irr}\, KB_i$

We say that an ordinary irreducible character $\chi \in \mathrm{Irr}\, KG$ belongs to a block B if $\chi \in \mathrm{Irr}\, KB$

def: the principal block B_0 is the unique block through which the map $\mathcal{O}G \twoheadrightarrow k$ factors
$$\sum \lambda_g g \longmapsto \sum \overline{\lambda_g}$$

Equivalently B_0 is the unique block such that $1_G \in \mathrm{Irr}_K B$

Ex: $G = \mathfrak{S}_3$, the partition into blocks when $\ell = 2$
is $\mathrm{Irr}_K \mathfrak{S}_3 = \underbrace{\{\chi_{\tiny\square\square\square}, \chi_{\tiny\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}}\}}_{\text{principal block}} \sqcup \{\chi_{\tiny\begin{smallmatrix}\square\square\\\square\end{smallmatrix}}\}$

Let $\chi \in \mathrm{Irr}_K G$ and $e_\chi = \dfrac{\chi(1)}{|G|} \sum_{g \in G} \chi(g) g^{-1}$

if $\ell \nmid \dfrac{|G|}{\chi(1)}$ then $e_\chi \in \mathcal{O}G$ and $\mathcal{O}Ge_\chi \simeq \mathrm{Mat}_{\chi(1)}(\mathcal{O})$
is a block with trivial defect and $\mathrm{Irr}_K B = \{\chi\}$

More generally to any block B one can attach an ℓ-subgroup D of G called a defect group of B

Ex: defect groups of principal blocks are Sylow subgroups

3) Principal blocks for finite reductive groups [Broué-Malle-Michel]

G connected reductive group / $\overline{\mathbb{F}}_p$
$F : G \longrightarrow G$ Frobenius endomorphism
T quasi-split, $W = N_G(T)/T$

Fact: if $\ell \neq p$ and $\ell > h$ Sylow subgroups of G^F are abelian
$\Rightarrow$ Sylow ℓ-subgroups are contained in tori

Thm: Let D be a Sylow ℓ-subgroup of G^F and
$w \in W$ be such that $D \hookrightarrow T^{wF}$
Assume $C_G(D)$ is a maximal torus. Then
the unipotent characters in the principal block are
the consituents of the DL character R_w

Ex: $G = GL_n$ and $\ell > n$
* If $\ell \mid q-1$ then $T^F \supseteq \ell$-Sylow and
  the principal blocks contains all the unipotent char.
* If $\ell \mid \Phi_n(q)$ then $T^{(1,2,\dots,n)F} \simeq \mathbb{F}_{q^n}^\times \supseteq \ell$-Sylow
and the unipotent char. in the principal block are

$$1_{G^F} = \rho_{(n)}, \rho_{(n-1,1)}, \rho_{(n-2,1,1)}, \dots, \rho_{(1^n)} = St_{G^F}$$

Rmk: more generally $C_G(D)$ is a Levi subgroup and
one should consider parabolic version of DL characters