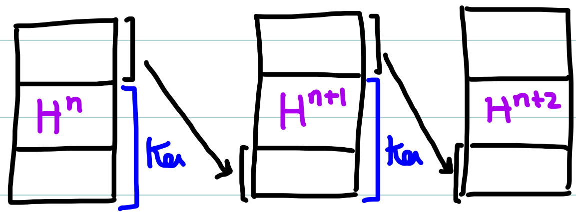
V-2 DERIVED CATEGORIES AND DERIVED EQUIVALENCES

1) - Toolbox for derived categories

$k = \bar{k}$ field and A finite dimensional k-algebra
A-mod = category of f.g A-modules

A complex of A-modules is

$$\cdots \longrightarrow C_n \xrightarrow{d_n} C_{n+1} \xrightarrow{d_{n+1}} C_{n+2} \longrightarrow \cdots \qquad \text{with } d_{n+1} \circ d_n = 0$$



The cohomology of $C_\bullet$ is $H^n(C_\bullet) = \mathrm{Ker}\, d_n / \mathrm{Im}\, d_{n-1}$

$C_\bullet$ can be shifted by any integer $r \in \mathbb{Z}$ by $(C_\bullet[r])_i = C_{i+r}$
An A-module M can be seen as a complex $M[r]$
where M is in degree $-r$ and all the other terms are zero

The (bounded) derived category $D^b(A)$ has
- objects: complexes of A-modules $C_\bullet$ s.t. $H^n(C_\bullet) = 0$ for $|n| \gg 0$
- morphisms: ???

Examples of morphisms

commuting with differentials

(a) If $f: C_\bullet \to D_\bullet$ is a morphism of complexes
which induces isomorphisms $H^\bullet(C_\bullet) \xrightarrow{\sim} H^\bullet(D_\bullet)$
(i.e a quasi-isomorphism)
$\Rightarrow$ f induces an isomorphism in $D^b(A)$

In particular a projective resolution

$$\cdots \to P_{-n-1} \to P_{-n} \to \cdots \to P_0 \twoheadrightarrow M$$

$P_\bullet$ is only bounded below

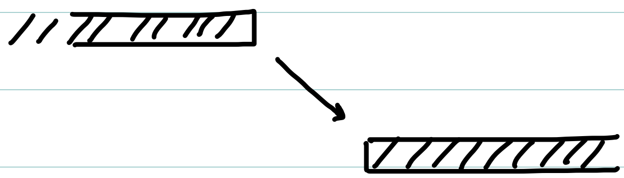
yields an isomorphism $P_\bullet \xrightarrow{\sim} M[0]$ in $D^-(A)$

(b) $\mathrm{Hom}_{D^b}(M[0], N[0]) \simeq \mathrm{Hom}_A(M,N)$

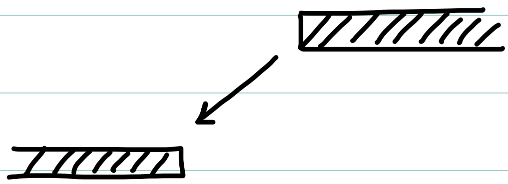
$\mathrm{Hom}_{D^b}(M[0], N[n]) \simeq \mathrm{Ext}^n_A(M,N)$ ($=0$ if $n<0$)

$\rightsquigarrow$ $D^b(kG)$ encodes the cohomology $H^\bullet(G,k)$ of G

More generally there are no non-zero maps



But there are non-zero maps



(c) If $H^n(C) = 0 \ \forall n$ then $C \to 0$ is an iso. in $D^b(A)$

(d) if $H^n(C) = 0$ except for one n, say $n = 0$
then $C_\bullet \simeq H^0(C)[0]$ [even though there are no maps of complexes $C_\bullet \to H^0(C)[0]$ or $H^0(C)[0] \to C_\bullet$ inducing an iso in general]

$$\begin{tikzcd}
\cdots & C_{-1} \arrow[r, "d_{-1}"] \arrow[d] & C_0 \arrow[r, "d_0"] \arrow[d, two heads] & C_1 \arrow[r] \arrow[d] & \cdots \\
\cdots & 0 \arrow[r] & C_0/\mathrm{Im}\, d_{-1} \arrow[r] & C_1 \arrow[r] & \cdots \\
\cdots & 0 \arrow[r] \arrow[u] & H^0(C) \arrow[r] \arrow[u, hook] & 0 \arrow[r] \arrow[u] & \cdots
\end{tikzcd}$$

(e) It stops here: $C_\bullet \not\simeq \bigoplus H^n(C)[-n]$ in general

if $H^n(C) = 0$ for $n \neq 0, r$ $(r > 0)$
then $C_\bullet$ is determined by

* The A-modules $H^0(C)$ and $H^r(C)$
* an elt of $\mathrm{Ext}_A^{r+1}(H^r(C), H^0(C))$

$\rightsquigarrow$ in $D^b(A)$ a complex C is encoded by $H^\bullet(C)$ and extensions between $H^i(C)$ and $H^j(C)$ $i \neq j$

2) Derived equivalences

A, B f.d algebras over $k=\bar{k}$
$\mathrm{Irr}A$, $\mathrm{Irr}B$ the isomorphism classes of simple modules

- Morita: $A\text{-mod} \xrightarrow{\sim} B\text{-mod}$
  $\Leftrightarrow$ $\exists$ B-module T such that:
  - T is projective
  - Every PIM of B is a direct summand of T] progenerator
  - $\mathrm{End}_B(T) \simeq A$

Then $T \otimes_A - : A\text{-mod} \xrightarrow{\sim} B\text{-mod}$ equivalence

- Derived: $D^b(A\text{-mod}) \xrightarrow{\sim} D^b(B\text{-mod})$
  $\Leftrightarrow$ $\exists$ complex of B-modules s.t
  - T is bounded, with projective terms [perfect complex]
  - Every perfect complex is obtained from T by direct sums, direct summands and cones
  - $\mathrm{End}_{D^b(B)}(T) \simeq A$ and $\mathrm{Hom}_{D^b(B)}(T, T[i]) = 0$ if $i \neq 0$

    $\hookrightarrow$ tilting complex

Then "$T \overset{L}{\otimes}_A -$" $: D^b(A\text{-mod}) \xrightarrow{\sim} D^b(B\text{-mod})$ equivalence

Rmk : if T is a perfect complex (bounded with projective terms) such that $T \simeq H^0(T)$ in $D^b(B\text{-mod})$ then $H^0(T)$ is not necessarily projective.
In that case T is tilting $\iff \forall i > 0 \ \mathrm{Ext}^i_B(T,T) = 0$
$\longrightarrow$ Tilting module

Rmk : $\# \mathrm{Irr} A = \mathrm{rk}_{\mathbb{Z}} K_0(A\text{-mod}) = \mathrm{rk}_{\mathbb{Z}} K_0(D^b(A))$ is preserved under derived equivalences.

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