V-3 BROUÉ'S ABELIAN DEFECT GROUP CONJECTURE

1) The conjecture for finite groups

$(K, \mathcal{O}, k)$ ℓ-mod. system

Let $k = \overline{\mathbb{F}_\ell}$ and G be a finite group
Given D a Sylow ℓ-subgroup of G :

(i) The restriction map $H^*(G,k) \longrightarrow H^*(D,k)$ is surjective.

(ii) If D is abelian the restriction map induces an isomorphism

$$H^*(G,k) \xrightarrow{\sim} H^*(N_G(D),k) \qquad \text{[Mislin]}$$

$H^*(G,k) = \mathrm{Ext}^*_{kG}(k,k)$ can be computed in the derived category of the principal block
(recall that $\mathrm{Ext}^i_{kG}(k,k) = \mathrm{Hom}_{D^b(kG)}(k, k[i])$)

Generalization to other representations and other blocks:

works with $\mathcal{O}$

Conj: let B be a block of kG with abelian defect D
b be the Brauer correspondant for $N_G(D)$
Then $D^b(B\text{-mod}) \simeq D^b(b\text{-mod})$

<u>Rmk</u>: D is a Sylow ℓ-subgroup when B is the principal block

$\Rightarrow H^i(G,k) \simeq H^i(N_G(D),k)$

But the restriction functor does not induce a desired equivalence in general!

BADGC was proved in the following cases:

- G ℓ-solvable
- D cyclic
- $D \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$
- $G = GL_n(q)$ and $\mathfrak{S}_n$

+ few other cases

<u>Problem</u>: finding suitable tilting complexes

2) <u>Consequences of the conjecture</u>

Recall that $\# \mathrm{Irr}\, B = \mathrm{rk}_{\mathbb{Z}} K_0(B\text{-mod})$

When BADGC holds, one gets

$$\# \mathrm{Irr}\, kB = \# \mathrm{Irr}\, kb \quad \text{(Brauer characters)}$$
$$\# \mathrm{Irr}\, KB = \# \mathrm{Irr}\, KB \quad \text{(ordinary characters)}$$

↑ actually a bijection with signs

+ numerical consequences depending on the version of the conj.

$\underline{\text{Ex}}$: $G = \mathcal{A}_5 = SL_2(\mathbb{F}_4)$ $\qquad N_G(5\text{-Sylow}) = D_5$

	1	$(12)(34)$	(123)	(12345)	(12354)
1	1	1	1	1	1
χ_3	3	-1	$\cdot$	α	$\bar\alpha$
χ_3'	3	-1	$\cdot$	$\bar\alpha$	α
χ_4	4	$\cdot$	1	-1	-1
χ_5	5	$\cdot$	1	$\cdot$	1

	1	s	r	r^2
1	1	1	1	1
$-\gamma_2$	-2	$\cdot$	$+\alpha$	$+\bar\alpha$
$-\gamma_2'$	-2	$\cdot$	$+\bar\alpha$	$+\alpha$
$-\gamma_1$	-1	$+1$	-1	-1

with $\alpha = \dfrac{1+\sqrt{5}}{2}$ and $\bar\alpha = \dfrac{1-\sqrt{5}}{2}$

3) Finite reductive groups

Now G connected reductive group $/ \overline{\mathbb{F}}_p$
and $F : G \longrightarrow G$ Frobenius
We assume $K \supseteq \overline{\mathbb{Q}}_\ell$ with $\ell \neq p$

For constructing ordinary characters of G^F we used the cohomology groups $H^i_c(\tilde{X}(w), K)$

There are versions over $\mathbb{Z}_\ell$ and $\mathbb{F}_\ell$ but they encode limited homological information for the representations of G^F

<u>Better</u> : a bounded complex of $(\mathcal{O}G^F, \mathcal{O}T^{wF})$-bimodules $R\Gamma_c(\tilde{X}(w), \mathcal{O})$

such that

- $H^i_c(\tilde{X}(w), \Lambda) \simeq H^i(R\Gamma_c(\tilde{X}(w), \mathcal{O}) \overset{L}{\otimes}_{\mathcal{O}} \Lambda)$
  for any ring $\Lambda \in \{K, \mathcal{O}, k\}$

- The terms of $R\Gamma_c(\tilde{X}(w), \mathcal{O})$ are finitely generated and projective as $\mathcal{O}G^F$ and $\mathcal{O}T^{wF}$-modules

- $R\Gamma_c(\tilde{X}(w), \mathcal{O})$ is well defined up to quasi-isomorphism

$\Rightarrow$ triangulated functor

$$\begin{array}{rcl} D^b(\mathcal{O}T^{wF}) & \longrightarrow & D^b(\mathcal{O}G^F) \\ C^\bullet & \longmapsto & R\Gamma_c(\tilde{X}(w), \mathcal{O}) \overset{L}{\otimes}_{\mathcal{O}T^{wF}} C^\bullet \end{array}$$

Let D be a Sylow ℓ-subgp of G^F
T' max torus s.t $D \subseteq T'^F$ (exists if $\ell > h$)
$\rightsquigarrow$ abelian defect !

$(T', F) \simeq (T, wF)$ for some $w \in W$
Assume $C_G(D) = T'$

$$\left.\begin{array}{l} N_{G^F}(D) = N_{G^F}(T') \\ C_{G^F}(D) = T'^F \end{array}\right] W(T')^F \longleftrightarrow W^{wF} \left[\begin{array}{l} N_{G^{wF}}(T) \\ T^{wF} \end{array}\right.$$

Conj: if $C_G(\ell\text{-Sylow})$ is a max torus of G then

$\exists\, w \in W$ s.t

(i) The action of T^{wF} on $R\Gamma_c(\widetilde{X}(w), \mathcal{O})$ extends to an action of $N_{G^{wF}}(T) = N$

(ii) $R\Gamma_c(\widetilde{X}(w), \Lambda) \overset{L}{\underset{N}{\otimes}} -$ induces a derived equivalence between the principal blocks of G^F and N^F

Rmk: in general $C_G(\ell\text{-Sylow})$ is a levi subgroup and one needs to consider parabolic versions of DL varieties

We have seen that the action of T^{wF} extends to an action of $T^{wF} \rtimes C_{B_w}(wF)$. It is expected that the map

$$\mathcal{O}T^{wF} \rtimes C_{B_w}(wF) \longrightarrow\!\!\!\!\!\rightarrow \mathrm{End}_{D^b(G^F)}\left(R\Gamma_c(\widetilde{X}(w))\right)$$

induces an isomorphism $\mathcal{O}N \xrightarrow{\sim} \mathrm{End}_{D^b(\mathcal{O}G^F)}\left(R\Gamma_c(\widetilde{X}(w))\right)$

Or some deformation of $\mathcal{O}T^{wF} \rtimes W^{wF}$