V-4 COHOMOLOGY COMPLEXES OF THE COXETER VARIETY

1) Cohomology groups over $\mathbb{Z}_\ell$ and $\mathbb{F}_\ell$

We assume that F acts trivially on W

As in II-4, let $c \in W$ be a Coxeter elt

Using the isomorphism $U_I^F \backslash X(c) \simeq X(c_I) \times (\mathbb{G}_m)^{|S \setminus I|}$
one can show:

Prop: if $\ell \nmid |G^F|$ then $H_c^\bullet(X(c), \mathbb{Z}_\ell)$ is torsion-free
Consequently $H_c^i(X(c), \mathbb{F}_\ell) = H_c^i(X(c), \mathbb{Z}_\ell) \otimes_{\mathbb{Z}_\ell} \mathbb{F}_\ell$

The case $\ell \mid |G^F|$ is more difficult

Thm: if $\ell \mid \Phi_h(q)$ and $\ell > h$ then $H_c^\bullet(X(c), \mathbb{Z}_\ell)$ is t.f.

Rmk: this last property is very specific to c
In general $H_c^\bullet(X(w), \mathbb{Z}_\ell)$ is not torsion-free when $\ell \mid |T^{wF}|$

These two results can be generalized from $X(c)$ to $\tilde{X}(c)$

2) Cohomology complexes

We now assume that $\ell \mid \Phi_h(q)$ and $\ell > h$

The cohomology complex $R\Gamma_c(X(c), \mathbb{Z}_\ell)$ is not perfect (the terms cannot all be projective $\mathbb{Z}_\ell G^F$-modules) but $R\Gamma_c(\tilde{X}(c), \mathbb{Z}_\ell)$ is !

We work with an intermediate variety

$$\begin{tikzcd} \tilde{X}(c) \arrow[r, two heads] \arrow[rr, bend right, "/T^{cF}"'] & X_\ell \arrow[r, two heads] & X(c) \end{tikzcd} \quad \text{s.t. } X_\ell = \tilde{X}(c)/(T^{cF})_{\ell'}$$

We have $R\Gamma_c(X_\ell, \mathbb{Z}_\ell) \simeq R\Gamma_c(\tilde{X}(c), \mathbb{Z}_\ell) \otimes_{\mathbb{Z}_\ell T^{cF}} \mathbb{Z}_\ell (T^{cF})_{\ell'}$

Since $\mathbb{Z}_\ell (T^{cF})_{\ell'}$ is a direct summand of $\mathbb{Z}_\ell T^{cF}$
$R\Gamma_c(X_\ell, \mathbb{Z}_\ell)$ is a direct summand of $R\Gamma_c(\tilde{X}(c), \mathbb{Z}_\ell)$
$\Rightarrow$ perfect and cohomology torsion-free.

- **Over K**:
$$\begin{aligned} R\Gamma_c(X_\ell, K) &\simeq \bigoplus H^i_c(X_\ell, K)[-i] \\ &\simeq \bigoplus H^i_c(\tilde{X}(c), K) \otimes_{T^{cF}} (T^{cF})_{\ell'} [-i] \\ &\simeq \bigoplus_{\theta \in \mathrm{Irr}(T^{cF})_\ell} \bigoplus H^i_c(\tilde{X}(c), K)_\theta [-i] \end{aligned}$$

$\theta \in \mathrm{Irr}(T^{cF})_\ell \iff \theta_{|(T^{cF})_{\ell'}} = 1$

The assumptions on ℓ $\Rightarrow$ if $\theta \in \mathrm{Irr}(T^{cF})_\ell \setminus \{1_{T^{cF}}\}$ then θ occurs only in the middle degree

Let $\chi_{exc} = \bigoplus (-1)^{\ell(c)} R_c(\theta)$ where θ runs over $\langle c \rangle$-orbits of $\mathrm{Irr}(T^{cF})_\ell \setminus \{1_{T^{cF}}\}$

Then $H_c^\bullet(X_\ell, K) = \underbrace{\chi_{exc}^{\oplus h}[-\ell(c)]}_{\text{non-unipotent part}} \oplus H_c^\bullet(X(c), K)$

- <u>Over k</u> : F has h eigenvalues $\lambda_1, \dots, \lambda_h$ on $H_c^\bullet(X(c), K)$ which are the h-th roots of 1 in k $(\supseteq \mathbb{F}_\ell)$

$\lambda \rightsquigarrow \rho_\lambda$ eigenspace of F on $H_c^\bullet(X(c))$
i_λ = degree in which it occurs

Let C_λ = λ-generalized eigenspace of F on $R\Gamma_c(X_\ell, \mathcal{O})$
Then
- C_λ is a perfect complex
- $H_c^\bullet(C_\lambda \otimes K) = \chi[-\ell(c)] \oplus \chi_\lambda[-i_\lambda]$

some direct summand of $\chi_{exc}^{\oplus h}$

From the theory of blocks with cyclic defects, characters of projective indec modules are $\chi_{exc} + \chi_\lambda$ or $\chi_\lambda + \chi_{\lambda'}$ $\lambda \neq \lambda'$

Consequently $\chi = \chi_{exc}$

From this one can determine C_λ and get information on the principal block

Ex: if $i_\lambda = \ell(c)$ then $C_\lambda \simeq P[-\ell(c)]$
where P is the unique PIM with character $\chi_{exc} + \chi_\lambda$

Example: $G = GL_n$, F standard Frobenius, $h = n$

$$\mathrm{Irr}\, kB_0 = \{\rho_{(n)}, \rho_{(n-1,1)}, \dots, \rho_{(1^n)}\} \sqcup \{R_c(\theta)\}_{\theta \in \mathrm{Irr}(T^{cF})_\ell \setminus \{1\}}$$

Projective modules have characters

- $\chi_{exc} + St \quad (= \chi_{exc} + \rho_{(1^n)}) \rightsquigarrow P_0$

or

- $\rho_{(i,1^{n-i})} + \rho_{(i+1,1^{n-i-1})} \rightsquigarrow P_i$

The eigenvalues of F are $1, q, \dots, q^{n-1}$ and

$$C_{q^i} \simeq 0 \to \underset{\substack{\uparrow \\ \text{in degree } n-1}}{P_0} \to P_1 \to \dots \to P_i \to 0$$

Prop: $\mathrm{Hom}_{D^b}(C_{q^i}; C_{q^j}[n]) = 0$ if $n \neq 0$

$\Rightarrow R\Gamma_c(X_\ell, k) = \bigoplus C_{q^i}$ is a tilting complex!

Rmk: same for other groups (see exercise for $Sp_4(q)$)

3) Endomorphism algebra

We have $C_{B_W}(c) = \langle c \rangle \simeq \mathbb{Z}$ $\circlearrowleft$ $R\Gamma_c(X_\ell)$ by F

$\rightsquigarrow$ morphism $T_\ell^{cF} \rtimes \langle F \rangle \longrightarrow \underbrace{\mathrm{End}_{D^b}(R\Gamma_c(X_\ell))}_{\text{dimension } |T_\ell^{cF}| \times h}$

Moreover all the eigenvalues of F are h-th roots of 1 in k
$\rightsquigarrow$ $F^h - 1$ is nilpotent
$\rightsquigarrow$ one can deform F into F' such that $F'^h = 1$
and $\mathrm{End}_{D^b}(R\Gamma_c(X_\ell)) \simeq T_\ell^{cF} \rtimes \langle F' \rangle / F'^h - 1$

$\Rightarrow$ the geometric version of Broué's conjecture holds
when $\ell \mid \Phi_h(q)$