Finite Reductive Groups

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1. Notation and Bibliographic Remarks

∘ *Notation*

Throughout $\mathbb{K} = \overline{\mathbb{F}_p}$ will be an algebraic closure of the finite field $\mathbb{F}_p$ of prime characteristic p. Recall that $\mathbb{K}$ is an algebraic group under addition, which we denote $\mathbb{K}^+$, and $\mathbb{K} \setminus \{0\}$ is an algebraic group under multiplication, which we denote $\mathbb{K}^\times$. Let $r = p^a$ be a power of p for some natural number $a > 0$ then we write $\mathbb{F}_r = \{x \in \mathbb{K} \mid x^r = x\}$ for the finite subfield of $\mathbb{K}$ of order r.

Affine varieties will typically be denoted by bold letters. If they are endowed with a generalised Frobenius endomorphism then the corresponding fixed point group will be denoted in roman letters. For example, if $\mathbf{X}$ is an affine variety and $F : \mathbf{X} \to \mathbf{X}$ is a generalised Frobenius endomorphism then X denotes the fixed point group $\mathbf{X}^F = \{x \in \mathbf{X} \mid F(x) = x\}$. Throughout $\mathbf{G}$ will typically be reserved for a connected reductive algebraic group. We will denote by $\mathbf{T}_0$ and $\mathbf{B}_0$ a fixed maximal torus and Borel subgroup of $\mathbf{G}$ such that $\mathbf{T}_0 \leqslant \mathbf{B}_0$.

∘ *Bibliographic remarks*

Most of the material presented here is standard and covered in numerous textbooks. We give here a small sample of where to find more information on the material in each section.

2. Everything concerning root systems and Coxeter groups is covered in the first two chapters of [Hum90]. For a more rewarding read one can also consult Chapter 6 of [Bou02]. The material on root data can be found in Chapter 4 of [Car93] or again, for a more rewarding read see Demazure's excellent articles in [GP11] (in particular Exposé XXI).

3. The theory of algebraic groups has been developed in many textbooks. Everything we mention here, except dual groups, is contained in the standard references [Gec03], [Hum75] and [Spr09]. The material on dual groups is covered in Chapter 4 of [Car93]. For those unfamiliar with algebraic geometry then [Gec03] is a good starting point followed by [Spr09] for more structural results. For more on Lie algebras one can consult [EW06] and [Hum78] (the latter going much further).

4. Everything we use here is contained in first part of Chapter 4 of [Gec03].

5. Everything mentioned here is covered in the standard references [Car93], [DM91] and [Lus84]. For Deligne–Lusztig characters one may also consult Chapter 4 of [Gec03], which is recommended for those unfamiliar with the theory. Most of the results we mention here were proved originally by Deligne and Lusztig, many of them in their landmark paper [DL76]. The Jordan decomposition was proved for connected centre groups in [Lus84] and was later extended to the disconnected centre case in [Lus88], (see also the discussion in section 15 of [CE04]).

It is also worthwhile noting that many of the results in Sections 2 to 4 are succinctly covered in [MT11]. Additionally, the survey articles contained in [CG98] also cover many the topics we discuss here.

2. Root Systems, Coxeter Groups and Root Data

∘ *Root Systems and Coxeter Groups*

We will assume that V is a real Euclidean vector space endowed with a positive definite symmetric bilinear form $(\cdot,\cdot) : V \times V \to \mathbb{R}$.

Definition 2.1. For any $\alpha \in V$ we say a linear map $s_\alpha \in \mathrm{GL}(V)$ is a *reflection along* α if $s_\alpha(\alpha) = -\alpha$ and s_α fixes pointwise the hyperplane $H_\alpha = \{v \in V \mid v \perp \alpha\}$ orthogonal to α.

Using the fact that $V = H_\alpha \oplus \mathbb{R}\alpha$ we can see that for any $\alpha \in V$ the reflection s_α is unique and given by the formula

$$s_\alpha(v) = v - \langle v, \alpha\rangle\alpha \qquad \text{where} \qquad \langle v, \alpha\rangle := \frac{2(v,\alpha)}{(\alpha,\alpha)} \tag{2.1}$$

for all $v \in V$.

Exercise 2.2. Define a map $\langle -, -\rangle : V \times V \to \mathbb{R}$ by setting $\langle u, v\rangle = \frac{2(u,v)}{(v,v)}$. Show that this is not bilinear.

Exercise 2.3. Show that for each $\alpha \in V$ the reflection $s_\alpha \in \mathrm{GL}(V)$ is contained in the orthogonal group $\mathrm{O}(V)$. In other words we have $(s_\alpha u, s_\alpha v) = (u, v)$ for all $u, v \in V$.

Exercise 2.4. Given $w \in \mathrm{O}(V)$ prove that for each $\alpha \in V$ we have $ws_\alpha w^{-1} = s_{w\alpha}$.

Exercise 2.5. Given two roots $\alpha, \beta \in \Phi$ show that $s_\alpha s_\beta = s_\beta s_\alpha$ if α and β are orthogonal (i.e. $(\alpha, \beta) = 0$).

Definition 2.6. We say $\Phi \subseteq V$ is a *root system* of V if the following conditions are satisfied.

(R1) Φ is finite, $0 \notin \Phi$ and V is the $\mathbb{R}$-span of Φ.

(R2) for every $\alpha \in \Phi$ we have $c\alpha \in \Phi$ (for any $c \in \mathbb{R}$) implies $c = \pm 1$.

(R3) for every $\alpha \in \Phi$ the reflection s_α preserves Φ.

(R4) for any $\alpha, \beta \in \Phi$ we have $s_\alpha(\beta) \in \beta + \mathbb{Z}\alpha$.

Remark 2.7. Throughout we will assume that our root systems are *cyrstallographic*, i.e. they satisfy the crystallographic condition (R4). In general one may drop this condition to obtain a wider class of root systems but we will not consider this here (see [Hum90]).

If $\Phi \subseteq V$ is a root system then we define $W_\Phi = \langle s_\alpha \mid \alpha \in \Phi \rangle \leqslant \mathrm{O}(V)$ to be the *Weyl group* of Φ. In general, we define a Weyl group to be any group isomorphic to W_Φ for some root system Φ. Although simple in nature, root systems form the underlying ingredient in the classification of more complicated Lie type objects such as semisimple Lie algebras and connected reductive algebraic groups.

We say two root systems Φ_1 and Φ_2 are *isomorphic* if there exists a vector space isomorphism $\varphi : \mathbb{R}\Phi_1 \to \mathbb{R}\Phi_2$ such that $\langle \varphi(\beta), \varphi(\alpha) \rangle = \langle \beta, \alpha \rangle$ for all roots $\beta, \alpha \in \Phi_1$. Note that an isomorphism of root systems induces a natural isomorphism $W_{\Phi_1} \to W_{\Phi_2}$ of the corrrsponding Weyl groups sending $s_\alpha \mapsto s_{\varphi(\alpha)}$ for all $\alpha \in \Phi_1$. However, not all isomorphisms between Weyl groups arise from isomorphisms of the underlying root systems. To classify root systems we need to introduce the notion of a simple system of roots, which is analogous to the notion of a basis for a vector space.

Definition 2.8. Assume $\leqslant$ is a total ordering on V, then we say $v \in V$ is *positive* (with respect to $\leqslant$) if $0 \leqslant \lambda$. If $\Phi \subseteq V$ is a root system then we call

$$\Phi^+ := \{\alpha \in \Phi \mid 0 \geqslant \alpha\}$$

a *system of positive roots* or *positive system* (defined by $\leqslant$).

Exercise 2.9. Show that positive systems exist for any root system by defining a total ordering on any Euclidean vector space V.

Exercise 2.10. Assume $\Phi^+ \subset \Phi$ is a positive system then define $\Phi^- := \{-\alpha \mid \alpha \in \Phi^-\}$ to be the corresponding *negative system*. Check that Φ is a disjoint union $\Phi^+ \sqcup \Phi^-$.

Definition 2.11. Let $\Phi \subseteq V$ be a root system. We say $\Delta \subset \Phi$ is a *system of simple roots* or *simple system* if Δ is a basis of V and each $\alpha \in \Phi$ is a linear combination of elements from Δ where all coefficients are of the same sign (i.e. all positive or negative).

In opposition to positive systems, it is not clear from the definition that simple systems exist for arbitrary root systems. However, this is the case.

Theorem 2.12. *Assume $\Phi \subseteq V$ is a root system then the following hold.*

(i) Every positive system $\Phi^+ \subset \Phi$ contains a unique simple system, hence simple systems exist.

(ii) Any simple system $\Delta \subset \Phi$ is contained in a positive system.

(iii) The Weyl group W_Φ acts simply transitively on the set of positive systems (hence simple systems) of Φ.

(iv) Assume $\Delta \subset \Phi$ is a simple system and let $\mathbb{S} = \{s_\alpha \mid \alpha \in \Delta\}$ be the corresponding set of reflections then

$$W_\Phi = \langle s_\alpha \in \mathbb{S} \mid (s_\alpha s_\beta)^{m(\alpha,\beta)} = 1 \rangle,$$

where $m(\alpha, \beta)$ is the order of the product $s_\alpha s_\beta$.

(v) Given any $\beta \in \Phi$ there exists $w \in W_\Phi$ such that $w\beta \in \Delta$, where Δ is a fixed simple system of Φ.

Remark 2.13. Note that (v) in Theorem 2.12 says that a root system Φ can be recovered from a simple system Δ by acting with elements of the Weyl group.

The above theorem shows that the Weyl group W_Φ is an example of a wider class of groups known as *Coxeter groups*. We note that Coxeter groups that are not Weyl groups can be obtained from root systems by relaxing some of the conditions in Definition 2.6, for example the crystallographic condition (R4).

Definition 2.14. Assume W is a group and $\mathbb{S} \subseteq W$ is a finite generating set such that W has a presentation given by

$$W = \langle s \in \mathbb{S} \mid (st)^{m_{st}} = 1 \rangle$$

for some $m_{st} \in \mathbb{Z} \cup \{\infty\}$. We say $(W, \mathbb{S})$ a *Coxeter system* and W a *Coxeter group* if $m_{ss} = 1$ for all $s \in \mathbb{S}$ and $m_{st} \geqslant 2$ for all $s \neq t \in \mathbb{S}$.

Exercise 2.15. If $(W, \mathbb{S})$ is a Coxeter system prove that $m_{st} = m_{ts}$ for all $s, t \in \mathbb{S}$. In particular, the matrix $(m_{st})_{s,t\in\mathbb{S}}$ is symmetric.

If Δ is a fixed simple system of Φ then we call $\mathbb{S} = \{s_\alpha \mid \alpha \in \Delta\}$ the corresponding set of *simple reflections*. By (iv), we have for every $w \in W_\Phi$ that there exists $s_1, \dots, s_r \in \mathbb{S}$ such that $w = s_1 \cdots s_r$. We define the *length* $\ell(w)$ of w to be the smallest r for which such an expression exists and call the corresponding expression *reduced* (note that reduced

$\langle \alpha, \beta \rangle$	$\langle \alpha, \beta \rangle$	θ	$\lvert\beta\rvert^2 / \lvert\alpha\rvert^2$
0	0	$\pi/2$	undetermined
1	1	$\pi/3$	1
-1	-1	$2\pi/3$	1
1	2	$\pi/4$	2
-1	-2	$3\pi/4$	2
1	3	$\pi/6$	3
-1	-3	$5\pi/6$	3

Table 2.1: Ratios of root lengths.

expressions are not unique!). This defines a function $\ell : W_\Phi \to \mathbb{N}$ called the length function (by convention we set $\ell(1) = 0$).

Exercise 2.16. Prove that the length function does not depend upon the set of simple reflections used to define it.

One could naturally ask what the maximum possible length of an element in W_Φ is and also which elements can attain this length. This is answered by the following lemma.

Lemma 2.17. *Assume Φ is a root system with positive system $\Phi^+ \subset \Phi$ and simple system $\Delta \subset \Phi^+$. For any $w \in W_\Phi$ we have $\ell(w) \leqslant |\Delta|$. Furthermore, there exists a unique element $w_0 \in W_\Phi$ satisfying $\ell(w_0) = |\Delta|$. We call w_0 the* longest element *of W_Φ*

Exercise 2.18. Prove that $w_0^2 = 1$.

If Φ is a root system and $\Delta \subset \Phi$ is a simple system then $|\Delta|$ is an invariant of the root system called the *rank* of the root system (this follows from Theorem 2.12). We now investigate the crystallographic condition (R4) in Definition 2.6, which we will see is surprisingly restrictive. Firstly, let us note that (R4) is equivalent to the statement $\langle \beta, \alpha \rangle \in \mathbb{Z}$ for all $\alpha, \beta \in \Phi$. If θ is the angle between two roots $\alpha, \beta \in \Phi$ then the cosine of the angle θ is given by the formula

$$(\alpha, \beta) = |\alpha| \cdot |\beta| \cos\theta$$

where $|\alpha|$ denotes the length of the root. In particular we obtain

$$\langle \beta, \alpha \rangle = \frac{2(\beta, \alpha)}{(\alpha, \alpha)} = 2\frac{|\beta|}{|\alpha|}\cos\theta \qquad \Rightarrow \qquad \langle \alpha, \beta \rangle \langle \beta, \alpha \rangle = 4\cos^2\theta,$$

which implies $\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle$ is a non-negative integer.

Exercise 2.19. Assume $\alpha, \beta \in \Phi$ are two roots such that $\alpha \neq \pm\beta$ and $|\beta| \geqslant |\alpha|$. Prove that Table 2.1 gives all possibilities for the values $\langle \alpha, \beta \rangle$, $\langle \beta, \alpha \rangle$, θ and $|\beta|^2/|\alpha|^2$ (where θ is the angle between α and β). (Hint: use the fact that $0 \leqslant \cos^2\theta \leqslant 1$ and $\langle \alpha, \beta \rangle$, $\langle \beta, \alpha \rangle$ have like

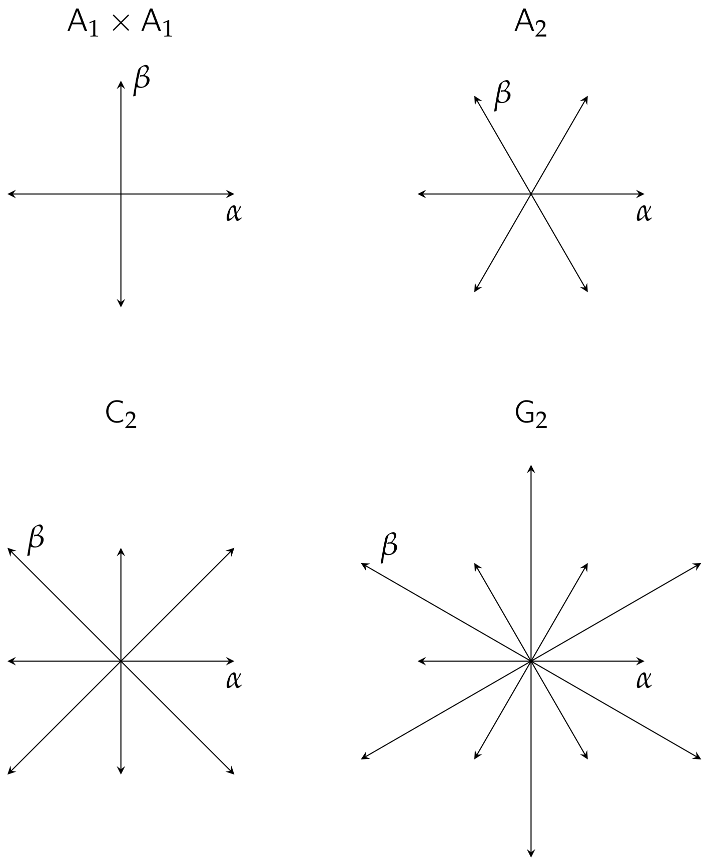


Figure 1: Rank 2 Root Systems

sign.)

Exercise 2.20. Check that each collection of vectors (in $\mathbb{R}^2$) described in Figure 1 defines a root system and prove that these are the only root systems of rank 2 up to isomorphism. Here $\Delta = \{\alpha, \beta\}$ denotes a simple system for the root system $\Phi \subset \mathbb{R}^2$. Describe the Weyl groups of these root systems up to isomorphism.

We wish to now describe the classification of root systems. However, to do this we need to introduce the following notion.

Definition 2.21. A root system Φ is called *decomposable* (or *reducible*) if there exist proper non-empty subsets $\Phi_1, \Phi_2 \subset \Phi$ such that $\Phi = \Phi_1 \cup \Phi_2$ and $(\alpha, \beta) = 0$ for all $\alpha \in \Phi_1$ and $\beta \in \Phi_2$. Conversely we say Φ is *indecomposable* (or *irreducible*) if Φ is not decomposable.

Example 2.22. The root systems A_2, C_2 and G_2 of Exercise 2.20 are indecomposable, while the root system $\mathsf{A}_1 \times \mathsf{A}_1$ is decomposable.

Exercise 2.23. Prove that any decomposition $\Phi = \Phi_1 \cup \Phi_2$ of a root system Φ (as in Definition 2.21) is necessarily disjoint, i.e. $\Phi_1 \cap \Phi_2 = \varnothing$.

Exercise 2.24. Let Φ be a root system with simple system $\Delta \subset \Phi$. Show that Φ is indecomposable if and only if Δ cannot be written as a disjoint union $\Delta_1 \sqcup \Delta_2$ of two proper non-empty subsets such that $(\alpha, \beta) = 0$ for all $\alpha \in \Delta_1$ and $\beta \in \Delta_2$. (Hint: use Exercise 2.5 and Theorem 2.12).

Proposition 2.25. *Let $\Phi \subset V$ be a decomposable root system with basis Δ. By the previous two exercises there exists a decomposition $\Delta = \Delta_1 \sqcup \Delta_2$ of Δ such that Δ_1 and Δ_2 are disjoint. Let $V_i = \mathbb{R}\Delta_i$ and $\Phi_i = \Phi \cap V_i$ for $i \in \{1, 2\}$ then the following hold.*

(i) $\Phi_i \subset V_i$ is a root system with basis Δ_i for $i \in \{1, 2\}$ and $\Phi = \Phi_1 \sqcup \Phi_2$ is a decomposition of Φ.

(ii) $W_\Phi = W_{\Phi_1} \times W_{\Phi_2}$ where we consider any element of W_{Φ_i}, for $i \in \{1, 2\}$, as a reflection on V.

This proposition shows that to classify root systems and Coxeter groups it suffices to classify the indecomposable root systems and their corresponding Coxeter groups. The classification statement for indecomposable root systems is most effectively done through the notion of a *Dynkin diagram*.

Definition 2.26. Assume Φ is a root system with simple system Δ of cardinality n. Recall from Exercise 2.19 that for any two roots $\alpha, \beta \in \Phi$ we have $\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle \in \{0, 1, 2, 3\}$. We define the *Dynkin diagram* of Φ to be the graph having n-vertices, labelled by the elements of Δ, such that distinct $\alpha, \beta \in \Delta$ are joined by $\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle$ number of edges. Furthermore, if $\alpha, \beta \in \Delta$ are of different lengths then we affix an arrow to the diagram pointing towards the shorter of the two roots.

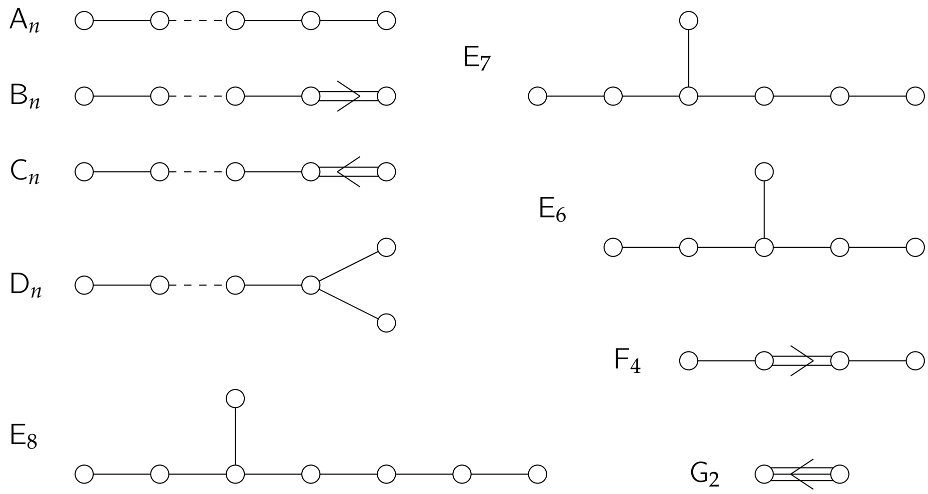
Example 2.27. The Dynkin diagrams of the rank 2 root systems given in Exercise 2.20 are as follows.

Exercise 2.28. Prove the Dynkin diagram is an invariant of a root system. In other words, it does not depend upon the choice of simple system used to define it. Furthermore, show that a root system is uniquely determined by its Dynkin diagram up to isomorphism.

Exercise 2.29. Show that a root system is indecomposable if and only if its Dynkin diagram is connected.

With the Dynkin diagram we may now give the elegant classification of indecomposable root systems.

Theorem 2.30. *If Φ is an indecomposable root system then its Dynkin digram is contained in the following list.*



Furthermore, every connected graph on this list arises as the Dynkin diagram of some indecomposable root system.

The indecomposable root systems are described in many places, for example in the plates of [Bou02]. We will not describe all such root systems here but we will describe the infinite family A_{n-1} as an example.

Example 2.31. Let $\{e_1, \dots, e_n\}$ be the standard basis of $\mathbb{R}^n$ then we have

$$\Phi = \{e_i - e_j \mid 1 \leqslant i, j \leqslant n \text{ and } i \neq j\}$$

is a root system of the vector space $V = \{(x_1, \dots, x_n) \mid \sum_{i=1}^{n} x_i = 0\} \subset \mathbb{R}^n$. A positive and simple system of roots for Φ may be given by

$$\begin{aligned} \Phi^+ &= \{e_i - e_j \mid 1 \leqslant i < j \leqslant n\} \\ \Delta &= \{e_i - e_{i+1} \mid 1 \leqslant i \leqslant n-1\}. \end{aligned}$$

For each $1 \leqslant i \leqslant n-1$ let α_i denote the simple root $e_i - e_{i+1}$. As $(e_i, e_j) = \delta_{ij}$ (the Kronecker delta) it is easy to check that

$$\langle \alpha_i, \alpha_j \rangle \langle \alpha_j, \alpha_i \rangle = \begin{cases} 0 & \text{if } |i-j| > 1, \\ 1 & \text{if } |i-j| = 1, \\ 4 & \text{if } i = j. \end{cases}$$

In particular, Φ is an indecomposable root system of type A_{n-1}.

For all $1 \leqslant i \leqslant n$ let us denote by $s_i \in W_\Phi$ the reflection corresponding to the simple root α_i. It is easy to check that for any $1 \leqslant k \leqslant n$ we have

$$s_i(e_k) = \begin{cases} e_{i+1} & \text{if } k = i \\ e_i & \text{if } k = i+1 \\ e_k & \text{if } k \notin \{i, i+1\}. \end{cases}$$

In particular, s_i acts on the standard basis as the transposition $(i, i+1)$. Therefore, W_Φ is isomorphic to the symmetric group $\mathfrak{S}_n$.

∘ *Root Data*

Definition 2.32. We say the quadruple $\Psi = (X, \Phi, \check{X}, \check{\Phi})$ is a *root datum* if the following conditions are satisfied.

(i) Both X and $\check{X}$ are free abelian groups of finite rank. Furthermore there exists a non-degenerate bilinear map $\langle -, - \rangle : X \times \check{X} \to \mathbb{Z}$ such that $\chi \mapsto \langle \chi, - \rangle$ and $\gamma \mapsto \langle -, \gamma \rangle$ give isomorphisms $X \to \mathrm{Hom}(\check{X}, \mathbb{Z})$ and $\check{X} \to \mathrm{Hom}(X, \mathbb{Z})$, (i.e. $\langle -, - \rangle$ is a *perfect pairing*).

(ii) Φ and $\check{\Phi}$ are finite subsets of X and $\check{X}$ respectively. Furthermore there exists a bijection $\Phi \to \check{\Phi}$ denoted by $\alpha \mapsto \check{\alpha}$, such that $\langle \alpha, \check{\alpha} \rangle = 2$.

(iii) For every $\alpha \in \Phi$ the maps $s_\alpha : X \to X$ and $s_{\check{\alpha}} : \check{X} \to \check{X}$ defined by

$$\begin{aligned} s_\alpha(\chi) &= \chi - \langle \chi, \check{\alpha} \rangle \alpha && \text{for all } \chi \in X, \\ s_{\check{\alpha}}(\gamma) &= \gamma - \langle \alpha, \gamma \rangle \check{\alpha} && \text{for all } \gamma \in \check{X}, \end{aligned}$$

are such that $s_\alpha(\Phi) = \Phi$ and $s_{\check{\alpha}}(\check{\Phi}) = \check{\Phi}$.

Exercise 2.33. Let $(X, \Phi, \check{X}, \check{\Phi})$ be a root datum. Show that the free abelian groups X and $\check{X}$ have the same finite rank.

Exercise 2.34. Show that $(X, \Phi, \check{X}, \check{\Phi})$ is a root datum if and only if $(\check{X}, \check{\Phi}, X, \Phi)$ is a root datum.

Assume now that $\Psi = (X, \Phi, \check{X}, \check{\Phi})$ is a root datum. As X, $\check{X}$ are abelian groups they are also $\mathbb{Z}$-modules hence we may form the tensor products of $\mathbb{Z}$-modules $\mathbb{R}X = \mathbb{R} \otimes_{\mathbb{Z}} X$ and $\mathbb{R}\check{X} = \mathbb{R} \otimes_{\mathbb{Z}} \check{X}$. The tensor products $\mathbb{R}X$ and $\mathbb{R}\check{X}$ are real vector spaces and we may identify X and $\check{X}$ with their natural images in these spaces. Typically we will suppress the tensors when writing elements of $\mathbb{R}X$ and $\mathbb{R}\check{X}$. We may easily extend the bilinear map in Definition 2.32 to a non-degenerate bilinear map $\mathbb{R}X \times \mathbb{R}\check{X} \to \mathbb{Z}$ by setting

$$\langle r_1\chi, r_2\gamma\rangle = r_1r_2\langle \chi, \gamma\rangle \qquad \text{for all } r_1, r_2 \in \mathbb{R}, \chi \in X \text{ and } \gamma \in \check{X}.$$

Extending linearly we may consider the automorphisms s_α, $s_{\check{\alpha}}$ of X and $\check{X}$ to be elements of $\mathrm{GL}(\mathbb{R}X)$ and $\mathrm{GL}(\mathbb{R}\check{X})$

Exercise 2.35. Check that the sub $\Phi \subseteq V$ and $\check{\Phi} \subseteq \check{V}$ are root systems where $V \subseteq \mathbb{R}X$ and $\check{V} \subseteq \mathbb{R}\check{X}$ are the subspaces spanned by Φ and $\check{\Phi}$.

Exercise 2.36. Construct a W_Φ-invariant positive definite symmetric bilinear form $(\cdot,\cdot) : \mathbb{R}X \times \mathbb{R}X \to \mathbb{R}$.

The map $s_\alpha \mapsto s_{\check{\alpha}}$ defines an isomorphism $W_\Phi \to W_{\check{\Phi}}$ between the corresponding Weyl groups and we call W_Φ the *abstract Weyl group* of the root datum. By the above exercise there exists a W_Φ-invariant positive definite symmetric bilinear form $(\cdot,\cdot) : \mathbb{R}X \times \mathbb{R}X \to \mathbb{R}$ which we assume fixed. The s_α are then Euclidean reflections with respect to this metric and are given by the formula in (2.1). In particular, for any root α we have $\langle -, \check{\alpha}\rangle = 2(\alpha,\alpha)^{-1}(-,\alpha)$. This form gives an identification of $\mathbb{R}\check{X}$ as the dual vector space of $\mathbb{R}X$. It follows that under this identification we have

$$\check{\alpha} = \frac{2\alpha}{(\alpha,\alpha)}, \tag{2.2}$$

(see [Bou02, Ch. VI - §1 - no. 1 - Lemma 2]).

Remark 2.37. Observe that the situation here is slightly more flexible than when we considered root systems. In particular, the identification $\langle -, \check{\alpha}\rangle = 2(\alpha,\alpha)^{-1}(-,\alpha)$ holds *only* for the given coroot $\check{\alpha} \in \check{\Phi}$ and does not make sense for other elements of $\check{X}$. Hence, the bilinearity of $\langle -, -\rangle$ does not contradict Exercise 2.2.

Example 2.38. Our principal example of a root datum $(X, \Phi, \check{X}, \check{\Phi})$ is the following. Let $e_1, \dots, e_n$ be the standard basis of $\mathbb{R}^n$ for some $n > 0$ then we set

$$\begin{aligned} X = \check{X} &= \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_n \subset \mathbb{R}^n, \\ \Phi = \check{\Phi} &= \{e_i - e_j \mid 1 \leqslant i, j \leqslant n \text{ and } i \neq j\}. \end{aligned}$$

The set Φ is the root system of type A_{n-1} described in Example 2.31. We define our non-degenerate bilinear map by setting $\langle e_i, e_j\rangle = \delta_{ij}$ for all $1 \leqslant i, j \leqslant n$ and extending linearly.

The bijection $\Phi \mapsto \check{\Phi}$ is simply the identity and it is readily checked that $\langle \alpha, \check{\alpha} \rangle = 2$ for all $\alpha \in \Phi$. We leave the verification of the third condition in Definition 2.32 as an exercise.

Let $\Psi' = (X', \Phi', \check{X}', \check{\Phi}')$ be another root datum and assume $\varphi \in \mathrm{Hom}(X', X)$ is a homomorphism of abelian groups. Precomposing with φ gives an induced homomorphism $\mathrm{Hom}(X, \mathbb{Z}) \to \mathrm{Hom}(X', \mathbb{Z})$. By identifying $\mathrm{Hom}(X, \mathbb{Z})$ and $\mathrm{Hom}(X', \mathbb{Z})$ with $\check{X}$ and $\check{X}'$, using the respective perfect pairings, we obtain a homomorphism $\langle -, \gamma \rangle \mapsto \langle -, \gamma \rangle \circ \varphi$.

Definition 2.39. For any $\varphi \in \mathrm{Hom}(X', X)$ we define $\check{\varphi} \in \mathrm{Hom}(\check{X}, \check{X}')$ to be the unique homomorphism satisfying $\langle -, \check{\varphi}(\gamma) \rangle = \langle -, \gamma \rangle \circ \varphi$. We call $\check{\varphi}$ the *dual* of φ. Equivalently we have $\check{\varphi}$ is the unique homomorphism satisfying $\langle \varphi(\chi), \gamma \rangle = \langle \chi, \check{\varphi}(\gamma) \rangle$ for all $\chi \in X'$ and $\gamma \in \check{X}$.

We have a similar duality map $\mathrm{Hom}(\check{X}, \check{X}') \to \mathrm{Hom}(X', X)$, also denoted $\varphi \mapsto \check{\varphi}$, and the composition of these dualities satisfies $\check{\check{\varphi}} = \varphi$.

Exercise 2.40. Prove that $\varphi \in \mathrm{Hom}(X', X)$ is surjective (resp. injective) if and only if its dual $\check{\varphi} \in \mathrm{Hom}(\check{X}, \check{X}')$ is injective (resp. surjective). In particular, φ is an isomorphism if and only if $\check{\varphi}$ is an isomorphism.

Remark 2.41. It is easily seen that the automorphism $s_{\check{\alpha}}$ of $\check{X}$ in Definition 2.32 is the automorphism dual to s_α, thus we may often denote this by $\check{s}_\alpha$.

Definition 2.42. We say $\varphi : \Psi' \to \Psi$ is an *isomorphism of root data* if the following conditions hold:

- $\varphi : X' \to X$ is an isomorphism of abelian groups.
- the restriction $\varphi|_{\Phi'}$ determines a bijection $\Phi' \to \Phi$ and $\check{\varphi}(\check{\alpha}) = \widecheck{\varphi(\alpha)}$ for all $\alpha \in \Phi'$.

Let us choose a positive system of roots $\Phi^+ \subset \Phi$ then, by Theorem 2.12, this determines a unique system of simple roots $\Delta \subset \Phi^+$. Let $\check{\Delta}$ be the image of Δ under the bijection $\Phi \to \check{\Phi}$ then by [Bou02, Ch. VI - §1 - no. 5 - Remark (5)] and (2.2) we have $\check{\Delta}$ is a system of simple roots for $\check{\Phi}$, hence $\check{\Delta}$ determines a unique system of positive roots $\check{\Phi}^+ \subset \check{\Phi}$.

We call the following subspaces of $\mathbb{R}X$ and $\mathbb{R}\check{X}$

$$\begin{aligned}
\Lambda &= \{\chi \in \mathbb{R}X \mid \langle \chi, \check{\alpha} \rangle \in \mathbb{Z} \text{ for all } \alpha \in \Phi\} \\
\check{\Lambda} &= \{\gamma \in \mathbb{R}\check{X} \mid \langle \alpha, \gamma \rangle \in \mathbb{Z} \text{ for all } \check{\alpha} \in \check{\Phi}\},
\end{aligned}$$

the *weight lattice* and *coweight lattice* of Ψ, (their elements are called weights and coweights respectively). If $\mathbb{R}X$ is the $\mathbb{R}$-span of Φ then we have the following sequence of subspace inclusions $\mathbb{Z}\Phi \subseteq X \subseteq \Lambda$ where $\mathbb{Z}\Phi$ is the $\mathbb{Z}$-span of the roots (similarly we have the inclusions $\mathbb{Z}\check{\Phi} \subseteq \check{X} \subseteq \check{\Lambda}$). The two quotient spaces

$$\Pi = \Lambda/\mathbb{Z}\Phi \qquad\qquad \check{\Pi} = \check{\Lambda}/\mathbb{Z}\check{\Phi}$$

have the structure of a finite abelian group and are isomorphic, (see [Bou02, Ch. VI - §1 - no. 9]). We call this group the *fundamental group* of Φ, (this should not be confused with Λ/X which is also called the fundamental group of Ψ).

Example 2.43. Let Ψ be the root datum of Example 2.38 whose root system Φ is of type A_{n-1}. The fundamental group Π of Φ is a cyclic group of order n.

Exercise 2.44. Let Ψ be as in Example 2.38. Prove that the fundamental group of Ψ (i.e. the quotient Λ/X) is non-trivial.

3. Algebraic Groups

∘ *Connected Reductive Algebraic Groups*

We will assume throughout that $\mathbf{G}$ is a non-empty algebraic set. By this, we mean there exists a set of polynomials $I(\mathbf{G}) \subseteq \mathbb{K}[X_1, \dots, X_n]$ such that

$$\mathbf{G} = \{(x_1, \dots, x_n) \in \mathbb{K}^n \mid f(x_1, \dots, x_n) = 0 \text{ for all } f \in I(\mathbf{G})\}.$$

The set of polynomials is an ideal, called the *vanishing ideal of* $\mathbf{G}$, which satisfies

$$I(\mathbf{G}) = \{f \in \mathbb{K}[X_1, \dots, X_n] \mid f(x) = 0 \text{ for all } x \in \mathbf{G}\}.$$

We will denote by $\mathbb{K}[\mathbf{G}]$ the affine algebra of $\mathbf{G}$ which is the quotient $\mathbb{K}[X_1, \dots, X_n]/I(\mathbf{G})$. We also recall that the set $\mathbf{G}$ is endowed with a topology called the *Zariski topology*.

Definition 3.1. If $\mathbf{G} \subseteq \mathbb{K}^n$ and $\mathbf{H} \subseteq \mathbb{K}^m$ are non-empty algebraic sets then we say $\varphi : \mathbf{G} \to \mathbf{H}$ is a *regular map* (or *morphism of varieties*) if there exist polynomials $f_1, \dots, f_m \in \mathbb{K}[X_1, \dots, X_n]$ such that

$$\varphi(x) = (f_1(x), \dots, f_m(x))$$

for all $x \in \mathbf{G}$. We say φ is an *isomorphism* if φ is bijective and its inverse φ^{-1} is also a regular map.

A particular special case is when $m = 1$ and $\mathbf{H} = \mathbb{K}$, hence $\varphi(x) = f(x)$ for some polynomial $f \in \mathbb{K}[X_1, \dots, X_n]$. Let $\bar{f} = f + I(\mathbf{G})$ be the residue class of f in the affine algebra $\mathbb{K}[\mathbf{G}]$ then the residue class of f is uniquely determined by φ. Conversely any residue class $\bar{f}$ gives rise to a unique morphism of varieties $\mathbf{G} \to \mathbb{K}$. In this way we may identify $\mathbb{K}[\mathbf{G}]$ with all regular maps $\mathbf{G} \to \mathbb{K}$, hence we sometimes call $\mathbb{K}[\mathbf{G}]$ the *algebra of regular functions*.

Given any regular map $\varphi : \mathbf{G} \to \mathbf{H}$ we have φ induces a $\mathbb{K}$-algebra homomorphism

$$\varphi^* : \mathbb{K}[\mathbf{H}] \to \mathbb{K}[\mathbf{G}]$$

given by $\varphi^*(\bar{g}) = \bar{g} \circ \varphi$. The assignment $\varphi \mapsto \varphi^*$ is contravariant. Before introducing the notion of an algebraic group we recall the following important result concerning regular maps.

Proposition 3.2. *Assume $\mathbf{G} \subseteq \mathbb{K}^n$ and $\mathbf{H} \subseteq \mathbb{K}^m$ are non-empty affine algebraic sets. Then the assignment $\varphi \mapsto \varphi^*$ defines a bijection*

$$\{\textit{regular maps } \mathbf{G} \to \mathbf{H}\} \xrightarrow{\sim} \{\mathbb{K}\textit{-algebra homomorphisms } \mathbb{K}[\mathbf{H}] \to \mathbb{K}[\mathbf{G}]\}$$

Furthermore, the following hold.

(i) φ *is* dominant *(i.e. $\overline{\varphi(\mathbf{G})} = \mathbf{H}$) if and only if φ^* is injective.*

(ii) φ *is a* closed embedding *(i.e. $\varphi(\mathbf{G}) \subseteq \mathbf{H}$ is closed and the restriction of φ to $\mathbf{G}$ defines an isomorphism $\mathbf{G} \to \varphi(\mathbf{G})$) if and only if φ^* is surjective.*

(iii) φ is an isomorphism if and only if φ^* is an isomorphism of $\mathbb{K}$-algebras.

Definition 3.3. We say $\mathbf{G}$ is an *affine algebraic group* if $\mathbf{G}$ is a non-empty algebraic set endowed with a group structure such that the multiplication and inversion maps

$$\begin{aligned} \mathbf{G} \times \mathbf{G} &\to \mathbf{G} \\ (x,y) &\to xy \end{aligned} \qquad\qquad \begin{aligned} \mathbf{G} &\to \mathbf{G} \\ x &\to x^{-1} \end{aligned}$$

are regular. Note that the topology on $\mathbf{G} \times \mathbf{G}$ is again the Zariski topology and *not* the product topology.

Remark 3.4. Such a group, as defined above, is often called a *linear algebraic group* because we have defined it using an embedding into affine space. The term affine algebraic group is typically reserved for the case where $\mathbf{G}$ is an abstract affine variety (see [Gec03, §2.1]). However, the two definitions are equivalent as for any affine algebraic group $\mathbf{G}$ there exists a closed embedding $\mathbf{G} \hookrightarrow \mathrm{GL}_n(\mathbb{K})$ for some n.

We will denote by $M_n(\mathbb{K})$ the set of all $n \times n$ matrices whose entries are in $\mathbb{K}$. We will consider this as an algebraic set by identifying it with the affine space $\mathbb{K}^{n^2}$, its affine algebra is simply $\mathbb{K}[X_{ij} \mid 1 \leqslant i,j \leqslant n]$.

Example 3.5. Recall the determinant polynomial defined by

$$\det = \sum_{\rho \in \mathfrak{S}_n} \operatorname{sgn}(\rho) X_{1\rho(1)} \cdots X_{n\rho(n)} \in \mathbb{K}[X_{ij} \mid 1 \leqslant i,j \leqslant n]$$

where $\mathfrak{S}_n$ is the symmetric group on n letters and $\operatorname{sgn} : \mathfrak{S}_n \to \{\pm 1\}$ is the sign character. With this in hand we may define our principal example of an affine algebraic group, namely the general linear group $\mathrm{GL}_n(\mathbb{K})$. To see that this is an algebraic set we identify $\mathrm{GL}_n(\mathbb{K})$ with

$$\{(A, y) \in M_n(\mathbb{K}) \times \mathbb{K} \mid y \det(A) - 1 = 0\}$$

under the natural projection map $(A, y) \mapsto A$. Specifically $\mathrm{GL}_n(\mathbb{K}) \subseteq \mathbb{K}^{n^2+1}$ is defined by the prime ideal $\langle Y \det -1 \rangle \trianglelefteq \mathbb{K}[X_{ij}, Y \mid 1 \leqslant i,j \leqslant n]$.

Example 3.6. A simpler example is also given by the special linear group

$$\mathrm{SL}_n(\mathbb{K}) = \{A \in M_n(\mathbb{K}) \mid \det(A) - 1 = 0\},$$

whose vanishing ideal is $\langle \det -1 \rangle \trianglelefteq \mathbb{K}[X_{ij} \mid 1 \leqslant i,j \leqslant n]$.

We now define some subgroups of an affine algebraic group $\mathbf{G}$ which play an important role in describing the structure of the group.

- The connected component $\mathbf{G}^\circ$ of $\mathbf{G}$ is the unique closed normal subgroup of $\mathbf{G}$ whose index $[\mathbf{G} : \mathbf{G}^\circ]$ is finite.
- The radical $R(\mathbf{G})$ of $\mathbf{G}$ is the unique maximal closed connected solvable normal subgroup of $\mathbf{G}$.
- The unipotent radical $R_u(\mathbf{G})$ of $\mathbf{G}$ is the unique maximal closed connected normal subgroup of $\mathbf{G}$, all of whose elements are unipotent.

With these subgroups to hand we may now make the following important definitions.

Definition 3.7. If $\mathbf{G}$ is an affine algebraic group then we say $\mathbf{G}$ is:

- *connected* if $\mathbf{G} = \mathbf{G}^\circ$ (or equivalently that it is connected in the Zariski topology),
- *reductive* if $R_u(\mathbf{G}) = \{1\}$,
- *semisimple* if $R(\mathbf{G}) = \{1\}$,
- *simple* if $\mathbf{G}$ is connected and contains no proper non-trivial closed connected normal subgroups.

Remark 3.8. Note that a simple algebraic group may contain a proper non-trivial closed normal subgroup but it must necessarily be finite. For example, the special linear group $\mathrm{SL}_n(\mathbb{K})$ is a simple algebraic group but the centre $Z(\mathrm{SL}_n(\mathbb{K}))$ is a non-trivial closed normal subgroup whenever n is not a power of p.

An affine algebraic group $\mathbf{H}$ is called a *torus* if it is isomorphic to a direct product $\mathbb{K}^\times \times \cdots \times \mathbb{K}^\times$ with a finite number of factors. It is called *unipotent* if all its elements are unipotent. With this in hand we may describe the internal structure of an affine algebraic group $\mathbf{G}$ by the following chain of normal subgroups.

$$\mathbf{G} \xrightarrow{\quad\text{finite}\quad} \mathbf{G}^\circ \xrightarrow{\quad\text{semisimple}\quad} R(\mathbf{G}) \xrightarrow{\quad\text{torus}\quad} R_u(\mathbf{G}) \xrightarrow{\quad\text{unipotent}\quad} \{1\}$$

Here the labels describe the quotients. For example $\mathbf{G}/\mathbf{G}^\circ$ is a finite group, $\mathbf{G}^\circ/R(\mathbf{G})$ is semisimple, etc. As the unipotent radical is contained in the radical we can see that any semisimple algebraic group is reductive but the converse statement is not true. The structure of a connected reductive algebraic group is even simpler than the above diagram suggests. In particular, we have the following result.

Proposition 3.9. *Assume $\mathbf{G}$ is a connected reductive algebraic group. Let $Z(\mathbf{G})$ be the centre and $[\mathbf{G}, \mathbf{G}]$ be the derived subgroup then the following hold:*

(i) $R(\mathbf{G}) = Z(\mathbf{G})^\circ$ *is a torus and* $[\mathbf{G}, \mathbf{G}]$ *is semisimple*

(ii) $R(\mathbf{G}) \cap [\mathbf{G}, \mathbf{G}]$ *is finite*

(iii) $\mathbf{G} = [\mathbf{G}, \mathbf{G}] \cdot Z(\mathbf{G})^\circ$ *(this is sometimes called an* almost direct product*).*

Example 3.10. Our principal example of a connected reductive algebraic group is the general linear group $\mathbf{G} = \mathrm{GL}_n(\mathbb{K})$. Its derived subgroup is the special linear group $\mathrm{SL}_n(\mathbb{K})$ and its centre is

$$R(\mathbf{G}) = Z(\mathbf{G}) = \{\lambda I_n \mid \lambda \in \mathbb{K}^\times\} \cong \mathbb{K}^\times,$$

where I_n is the $n \times n$ identity matrix. Note that the centre is connected and non-trival, so $\mathbf{G}$ is not semisimple. The intersection between the centre and the derived subgroup is

$$Z(\mathbf{G}) \cap [\mathbf{G}, \mathbf{G}] = \{\lambda I_n \mid \lambda \in \mathbb{K}^\times \text{ and } \lambda^n = 1\} \cong \mathbb{K}^\times,$$

which is a cyclic group of order $n/\gcd(p, n)$ (this is the centre of $\mathrm{SL}_n(\mathbb{K})$).

Let $\mathbf{H} \leqslant \mathbf{G}$ be a closed subgroup of an affine algebraic group $\mathbf{G}$ then we say $\mathbf{H}$ is a *maximal torus* of $\mathbf{G}$ if it is maximal amongst all subtori of $\mathbf{G}$ with respect to inclusion. We say $\mathbf{H}$ is a *Borel subgroup* if it is a maximal closed connected solvable subgroup of $\mathbf{G}$. The

Borel subgroups of a connected reductive algebraic group play a key role in describing the finer structure theory of such groups. We recall the following facts concerning Borel subgroups and maximal tori.

Proposition 3.11. *Let* $\mathbf{G}$ *be an affine algebraic group then the following hold.*

(i) Every maximal torus $\mathbf{G}$ *is contained in a Borel subgroup of* $\mathbf{G}$*.*

(ii) All Borel subgroups of $\mathbf{G}$ *are conjugate and furthermore all maximal tori of* $\mathbf{G}$ *are conjugate.*

(iii) If $\mathbf{G}$ *is connected then every Borel subgroup* $\mathbf{B}$ *is self-normalising, i.e.* $N_{\mathbf{G}}(\mathbf{B}) = \mathbf{B}$*.*

Example 3.12. Assume $\mathbf{G} = \mathrm{GL}_n(\mathbb{K})$ then the following are respectively a maximal torus and Borel subgroup of $\mathbf{G}$

$$\mathbf{T} = \left\{ \begin{bmatrix} \star & & 0 \\ & \ddots & \\ 0 & & \star \end{bmatrix} \right\} \qquad \mathbf{B} = \left\{ \begin{bmatrix} \star & \cdots & \star \\ & \ddots & \vdots \\ 0 & & \star \end{bmatrix} \right\}.$$

In other words $\mathbf{T}$ is the subgroup of diagonal matrices and $\mathbf{B}$ is the subgroup of upper triangular matrices.

We will assume from now on that $\mathbf{G}$ is a connected reductive algebraic group.

◦ *The Lie Algebra and the Roots*

By Proposition 3.11 we may fix a maximal torus $\mathbf{T}_0 \leqslant \mathbf{G}$ and a Borel subgroup $\mathbf{B}_0 \leqslant \mathbf{G}$ such that $\mathbf{B}_0$ contains $\mathbf{T}_0$. We wish to define the root datum of $\mathbf{G}$ relative to $\mathbf{T}_0$. As above, this will be a quadruple

$$\Psi(\mathbf{T}_0) = (X(\mathbf{T}_0), \Phi(\mathbf{T}_0), \check{X}(\mathbf{T}_0), \check{\Phi}(\mathbf{T}_0))$$

where $X(\mathbf{T}_0) = \mathrm{Hom}(\mathbf{T}_0, \mathbb{K}^\times)$ and $\check{X}(\mathbf{T}_0) = \mathrm{Hom}(\mathbb{K}^\times, \mathbf{T}_0)$ (recall that homomorphisms are those of algebraic groups). To define the sets $\Phi(\mathbf{T}_0) \subset X(\mathbf{T}_0)$ and $\check{\Phi}(\mathbf{T}_0) \subset \check{X}(\mathbf{T}_0)$ we will need to introduce the Lie algebra of $\mathbf{G}$.

Definition 3.13. Assume $\mathfrak{g}$ is a $\mathbb{K}$-vector space with a binary operation $[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$. We say $\mathfrak{g}$ is a *Lie algebra* (over $\mathbb{K}$) with *Lie bracket* $[-,-]$ if the following hold:

(i) $[\alpha x + \beta y, z] = \alpha[x,z] + \beta[y,z]$ and $[z, \alpha x + \beta y] = \alpha[z,x] + \beta[z,y]$ for all $x, y, z \in \mathfrak{g}$ and $\alpha, \beta \in \mathbb{K}$.

(ii) $[x,x] = 0$ for all $x \in \mathfrak{g}$.

(iii) $[x,[y,z]]+[z,[x,y]]+[y,[z,x]]=0$ for all $x,y,z\in\mathfrak{g}$.

Exercise 3.14. Check that the set of all matrices $M_n(\mathbb{K})$ is a Lie algebra with the Lie bracket given by $[x,y]=xy-yx$. We call this the *general linear Lie algebra* $\mathfrak{gl}_n(\mathbb{K})$.

In this subsection we wish to show that, in analogy with Lie groups, every affine algebraic group has a corresponding Lie algebra in its tangent space. With this in mind, we define for every point $p\in\mathbb{K}^n$ a map $d_p:\mathbb{K}[X_1,\dots,X_n]\to\mathbb{K}[X_1,\dots,X_n]$ called the *differential at* p by setting

$$d_p(f)=\sum_{i=1}^{n}\frac{\partial f}{\partial X_i}(p)X_i.$$

Assume $I(\mathbf{G})\trianglelefteq\mathbb{K}[X_{ij}\mid 1\leqslant i,j\leqslant n]$ is the vanishing ideal of $\mathbf{G}\subseteq M_n(\mathbb{K})$ then for any $x\in\mathbf{G}$ we define the *tangent space of* $\mathbf{G}$ *at* x to be

$$T_x(\mathbf{G})=\{v\in\mathbb{K}^n\mid d_x(f)(v)=0\text{ for all }f\in I(\mathbf{G})\}\subseteq M_n(\mathbb{K}).$$

Proposition 3.15.

(i) Let $1\in\mathbf{G}\subseteq M_n(\mathbb{K})$ be the identity element then the tangent space $T_1(\mathbf{G})\subseteq M_n(\mathbf{G})$ is a Lie subalgebra of $\mathfrak{gl}_n(\mathbb{K})$. In other words the Lie bracket $[x,y]=xy-yx$ defines a Lie algebra structure on $T_1(\mathbf{G})$. We call $\mathfrak{g}:=T_1(\mathbf{G})$ the Lie algebra of $\mathbf{G}$.

(ii) If $\varphi:\mathbf{G}\to\mathbf{H}$ is a homomorphism of algebraic groups then the differential $d_1(\varphi):T_1(\mathbf{G})\to T_1(\mathbf{H})$ is a homomorphism of Lie algebras (i.e. $d_1(\varphi)$ is a $\mathbb{K}$-linear map preserving the Lie bracket).

For any $x\in\mathbf{G}$ we denote by $\mathrm{Inn}_x:\mathbf{G}\to\mathbf{G}$ the corresponding inner automorphism defined by $\mathrm{Inn}_x(g)=xgx^{-1}$. Using Proposition 3.15 we have the corresponding differential $\mathrm{Ad}_x:=d_1(\mathrm{Inn}_x)$ is an automorphism of the Lie algebra. More generally the map $x\mapsto\mathrm{Ad}_x$ defines a *rational representation* of the algebraic group

$$\mathrm{Ad}:\mathbf{G}\to\mathrm{GL}(\mathfrak{g})$$

called the *adjoint representation* of $\mathbf{G}$. It is through the adjoint representation that we may identify the roots of the algebraic group $\mathbf{G}$ with respect to $\mathbf{T}_0$. Specifically for any $\alpha\in X(\mathbf{T}_0)$ we define the corresponding *weight space* on the Lie algebra to be

$$\mathfrak{g}_\alpha=\{v\in\mathfrak{g}\mid(\mathrm{Ad}\,t)(v)=\alpha(t)v\text{ for all }t\in\mathbf{T}_0\}\subseteq\mathfrak{g}.$$

It is clear to see that we obtain a vector space decomposition of the Lie algebra into its corresponding weight spaces, i.e. $\mathfrak{g}=\oplus_{\alpha\in X(\mathbf{T}_0)}\mathfrak{g}_\alpha$. We now have the set of roots is given by

$$\Phi(\mathbf{T}_0)=\{\alpha\in X(\mathbf{T}_0)\mid\alpha\neq 0\text{ and }\mathfrak{g}_\alpha\neq 0\}.$$

Example 3.16. Let $\mathbf{G} = \mathrm{GL}_n(\mathbb{K})$ then we will take $\mathbf{T}_0$ (resp. $\mathbf{B}_0$) to be the maximal torus of diagonal matrices (resp. Borel subgroup of upper triangular matrices) defined in Example 3.12. As one would expect the Lie algebra $\mathfrak{g}$ of $\mathrm{GL}_n(\mathbb{K})$ is simply the general linear Lie algebra $\mathfrak{gl}_n(\mathbb{K})$. To prove this one needs additional dimension arguments such as those used in [Hum75, §9.3 - Examples] which we will not discuss here. For each $1 \leqslant i, j \leqslant n$ we denote by E_{ij} the elementary matrix all of whose entries are 0 except the entry in the ith row and jth column which is 1. It is clear that we have the following decomposition of $\mathfrak{g}$ as a vector space

$$\mathfrak{gl}_n(\mathbb{K}) = \bigoplus_{i,j=1}^{n} \mathbb{K}E_{ij}, \tag{3.1}$$

where $\mathbb{K}E_{ij}$ dentoes the $\mathbb{K}$-span of E_{ij}.

We will denote by $e_i : \mathbf{T}_0 \to \mathbb{K}^\times$ the homomorphism given by $e_i(t) = t_i$ where $t \in \mathbf{T}_0$ is the diagonal matrix $\mathrm{diag}(t_1, \dots, t_n)$. These homomorphisms form a basis for $X(\mathbf{T}_0)$ as a free abelian group. According to [Hum75, §10.3 - Lemma A] we have $\mathrm{Ad}_x(y)$ is simply the matrix product xyx^{-1} for all $x \in \mathbf{G}$ and $y \in \mathfrak{gl}_n(\mathbb{K})$. Hence, for any diagonal matrix $t = \mathrm{diag}(t_1, \dots, t_n) \in \mathbf{T}_0$ we have

$$(\mathrm{Ad}\, t)(E_{ij}) = tE_{ij}t^{-1} = (t_i t_j^{-1})E_{ij} = (e_i - e_j)(t)E_{ij},$$

for all $1 \leqslant i, j \leqslant n$. It is clear that $e_i - e_j$ is non-zero whenever $i \neq j$ and, by the above calculation, the corresponding weight space $\mathfrak{g}_{e_i - e_j}$ is also non-zero as it contains E_{ij}.

For each $1 \leqslant i \leqslant n$ it is clear that we have E_{ii} is contained in the 0-weight space $\mathfrak{g}_0$ but as $E_{ij} \notin \mathfrak{g}_0$ whenever $i \neq j$ we must have

$$\mathfrak{g}_0 = \bigoplus_{i=1}^{n} \mathbb{K}E_{ii},$$

by comparing with the decomposition given in (3.1). In particular, this shows that the roots of $\mathbf{G}$ with respect to $\mathbf{T}_0$ are given by

$$\Phi(\mathbf{T}_0) = \{e_i - e_j \mid 1 \leqslant i, j \leqslant n \text{ and } i \neq j\}.$$

Hence, $\Phi(\mathbf{T}_0)$ is simply the root system of type A_{n-1} described in Example 2.31.

◦ *The Coroots and Chevalley's Classification Theorem*

In the previous subsection we have used the Lie algebra to define the roots of $\mathbf{G}$ with respect to $\mathbf{T}_0$. To complete the definition of the root datum of $\mathbf{G}$ we must give a definition of the set of coroots $\check{\Phi}(\mathbf{T}_0)$. To do this we need the following result.

Theorem 3.17. *Let $\alpha \in \Phi(\mathbf{T}_0)$ be a root then the following hold.*

(i) There exists a morphism $x_\alpha : \mathbb{K}^+ \to \mathbf{G}$ such that x_α is an isomorphism onto its image $\mathbf{X}_\alpha := \mathrm{Im}(x_\alpha)$ and $tx_\alpha(c)t^{-1} = x_\alpha(\alpha(t)c)$ for all $c \in \mathbb{K}^+$ and $t \in \mathbf{T}_0$. If $x' : \mathbb{K}^+ \to \mathbf{G}$ is another such isomorphism with these properties then there exists a unique $\lambda \in \mathbb{K}^\times$ such that $x'(c) = x_\alpha(\lambda c)$ for all $c \in \mathbb{K}^+$. In particular, $\mathbf{X}_\alpha$ is uniquely determined.

(ii) The subgroup $\langle \mathbf{X}_\alpha, \mathbf{X}_{-\alpha}\rangle \leqslant \mathbf{G}$ is isomorphic to either $\mathrm{SL}_2(\mathbb{K})$ or $\mathrm{PGL}_2(\mathbb{K})$. In particular, there exists a surjective homomorphism

$$\varphi_\alpha : \mathrm{SL}_2(\mathbb{K}) \to \mathbf{G}$$

such that for a suitable normalisation of x_α and $x_{-\alpha}$ we have

$$\varphi_\alpha \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix} = x_\alpha(c) \qquad \varphi_\alpha \begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix} = x_{-\alpha}(c) \qquad \varphi_\alpha \left\{ \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix} \,\middle|\, \lambda \in \mathbb{K}^\times \right\} \leqslant \mathbf{T}_0.$$

Furthermore

$$(\alpha \circ \varphi_\alpha)\left(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix} \right) = \lambda^2$$

for all $\lambda \in \mathbb{K}^\times$.

For each root $\alpha \in \Phi(\mathbf{T}_0)$ we call the subgroup $\mathbf{X}_\alpha \leqslant \mathbf{G}$ of Theorem 3.17 the *root subgroup* of α. With this in place we may now define the coroots of $\mathbf{G}$ with respect to $\mathbf{T}_0$. Specifically, given a root $\alpha \in \Phi(\mathbf{T}_0)$ we define the corresponding coroot $\check{\alpha} \in \check{\Phi}(\mathbf{T}_0)$ by setting

$$\check{\alpha}(\lambda) = \varphi_\alpha \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix} \in \mathbf{T}_0$$

for all $\lambda \in \mathbb{K}^\times$.

Although we have now defined all the ingredients in the root datum $\Psi(\mathbf{T}_0)$ we have not defined all the ingredients given in Definition 2.32. In particular, we define the non-degenerate pairing $\langle -,- \rangle : X(\mathbf{T}_0) \times \check{X}(\mathbf{T}_0) \to \mathbb{K}$ as follows. Given $\chi \in X(\mathbf{T}_0)$ and $\gamma \in \check{X}(\mathbf{T}_0)$ we have their composition $\chi \circ \gamma$ is contained in $\mathrm{Hom}(\mathbb{K}^\times, \mathbb{K}^\times)$. In particular there exists an integer $m_{\chi,\gamma} \in \mathbb{Z}$ such that $(\chi \circ \gamma)(\lambda) = \lambda^{m_{\chi,\gamma}}$. We then define $\langle -,- \rangle$ by setting $\langle \chi, \gamma \rangle = m_{\chi,\gamma}$.

Exercise 3.18. Prove that the quadruple $\Psi(\mathbf{T}_0) = (X(\mathbf{T}_0), \Phi(\mathbf{T}_0), \check{X}(\mathbf{T}_0), \check{\Phi}(\mathbf{T}_0))$ is a root datum (as defined in Definition 2.32) where the perfect pairing $\langle -,- \rangle : X(\mathbf{T}_0) \times \check{X}(\mathbf{T}_0) \to \mathbb{K}$ is defined as above.

Exercise 3.19. Assume $\mathbf{T} \leqslant \mathbf{G}$ is a maximal torus of $\mathbf{G}$ and let $\Psi(\mathbf{T})$ be the root datum of $\mathbf{G}$ defined with respect to $\mathbf{T}$. Prove that $\Psi(\mathbf{T})$ and $\Psi(\mathbf{T}_0)$ are isomorphic as root data (in the sense of Definition 2.42). In particular, up to isomorphism, the root datum $\Psi(\mathbf{T}_0)$ of $\mathbf{G}$

is independent of the choice of maximal torus $\mathbf{T}_0$. (Hint: use Proposition 3.11)

So far we have not used our fixed Borel subgroup $\mathbf{B}_0$ containing $\mathbf{T}_0$ in describing the roots of $\mathbf{G}$. However, we may use the Borel subgroup to obtain a positive system of roots by setting

$$\Phi^+(\mathbf{T}_0) = \{\alpha \in \Phi(\mathbf{T}_0) \mid \mathbf{X}_\alpha \leqslant \mathbf{B}_0\}.$$

By Theorem 2.12 this determines a unique set of simple roots $\Delta(\mathbf{T}_0)$ for $\Phi(\mathbf{T}_0)$. The following shows that, conversely, the Borel subgroup is determined by the system of positive roots

$$\mathbf{B}_0 = \langle \mathbf{T}_0, \mathbf{X}_\alpha \mid \alpha \in \Phi^+(\mathbf{T}_0)\rangle.$$

This gives us a correspondence between positive systems of $\Phi(\mathbf{T}_0)$ and Borel subgroups of $\mathbf{G}$ containing $\mathbf{T}_0$.

We call the quotient group $W_{\mathbf{G}}(\mathbf{T}_0) = N_{\mathbf{G}}(\mathbf{T}_0)/\mathbf{T}_0$ the *Weyl group* of $\mathbf{G}$ defined with respect to $\mathbf{T}_0$. Note that for any other choice of maximal torus $\mathbf{T}$ of $\mathbf{G}$ the Weyl groups $W_{\mathbf{G}}(\mathbf{T}_0)$ and $W_{\mathbf{G}}(\mathbf{T})$ are isomorphic (see Proposition 3.11). For each $w \in W_{\mathbf{G}}(\mathbf{T}_0)$ we assume $\dot{w} \in N_{\mathbf{G}}(\mathbf{T}_0)$ is a fixed representative of w and given $t \in \mathbf{T}_0$ we define t^w to be the element $\dot{w}^{-1}t\dot{w}$. This gives an action of $W_{\mathbf{G}}(\mathbf{T}_0)$ on $\mathbf{T}_0$ which induces an action of $W_{\mathbf{G}}(\mathbf{T}_0)$ on the character and cocharacter groups given by

$$\begin{aligned} (w \cdot \chi)(t) &= \chi(t^w) && \text{for all } \chi \in X,\ w \in \mathbf{W},\ t \in \mathbf{T}_0,\\ (\check{w} \cdot \gamma)(t) &= \gamma(\lambda)^w && \text{for all } \gamma \in \check{X},\ w \in \mathbf{W},\ \lambda \in \mathbb{K}^\times. \end{aligned}$$

Note that the following exercise justifies our notation.

Exercise 3.20. Prove that $\check{w} : \check{X}(\mathbf{T}_0) \to \check{X}(\mathbf{T}_0)$ is the dual of $w : X(\mathbf{T}_0) \to X(\mathbf{T}_0)$ in the sense of Definition 2.39.

Exercise 3.21. For each root $\alpha \in \Phi(\mathbf{T}_0)$ let us define

$$\dot{n}_\alpha = \varphi_\alpha \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Show that $\dot{n}_\alpha \in N_{\mathbf{G}}(\mathbf{T}_0)$.

Using Exercise 3.21 we obtain for each root $\alpha \in \Phi(\mathbf{T}_0)$ a corresponding element of the Weyl group $n_\alpha \in W_{\mathbf{G}}(\mathbf{T}_0)$ (i.e. by taking the image of $\dot{n}_\alpha$ in the quotient). With this we may give the following result relating the Weyl group of $\mathbf{G}$ with the abstract Weyl group of its underlying root datum.

Proposition 3.22. *Let $\mathbf{G}$ be a connected reductive algebraic group with maximal torus $\mathbf{T}_0$ and let $\Psi(\mathbf{T}_0)$ be the root datum of $\mathbf{G}$ defined with respect to $\mathbf{T}_0$. For each root $\alpha \in \Phi(\mathbf{T}_0)$ the action* *of the element n_α on both $X(\mathbf{T}_0)$ and $\check{X}(\mathbf{T}_0)$ stabilises the sets of roots and coroots respectively. Furthermore the map $n_\alpha \mapsto s_\alpha$ defines an isomorphism $W_{\mathbf{G}}(\mathbf{T}_0) \to W_{\Phi(\mathbf{T}_0)}$ between the Weyl group of $\mathbf{G}$ and the abstract Weyl group of its underlying root datum.*

One of the most startling things about connected reductive algebraic groups is that the relatively simple combinatorial data introduced in Section 2.2 completely determines the group up to isomorphism. More precisely, we have the following.

Theorem 3.23 (Chevalley). *The map $\mathbf{G} \mapsto \Psi(\mathbf{T}_0)$ determines a bijective correspondence*

$$\left\{\begin{array}{c}\textit{isomorphism classes of connected}\\ \textit{reductive algebraic groups over } \mathbb{K}\end{array}\right\} \longleftrightarrow \{\textit{isomorphism classes of root data}\}.$$

As we have seen above, the root subgroups play an important role in describing the structure of a connected reductive algebraic group. To make locating these subgroups slightly easier we recall the following lemma.

Lemma 3.24. *Assume $\alpha \in \Phi(\mathbf{T}_0)$ is a root then $\mathbf{S}_\alpha := (\operatorname{Ker}\alpha)^\circ$ is a proper subtorus of $\mathbf{T}_0$. If $\mathbf{G}_\alpha$ is the centraliser $C_{\mathbf{G}}(\mathbf{S}_\alpha)$ then the root subgroups $\mathbf{X}_\alpha$ and $\mathbf{X}_{-\alpha}$ are minimal 1-dimensional unipotent subgroups of $\mathbf{G}_\alpha$ normalised by $\mathbf{T}_0$.*

Example 3.25. In Example 3.16 we described the roots of $\mathrm{GL}_n(\mathbb{K})$. We now complete this example by describing the full root datum of $\mathbf{G} = \mathrm{GL}_n(\mathbb{K})$. Let $\alpha = e_i - e_j$ be the root of $\mathbf{G}$ described in Example 3.16 then a simple calculation shows that

$$\mathbf{S}_\alpha = \{\operatorname{diag}(t_1, \dots, t_n) \in \mathbf{T}_0 \mid t_i = t_j\} \qquad \mathbf{G}_\alpha = \left\{\begin{bmatrix} \star & & & \\ & \ddots & \star & \\ & \star & \ddots & \\ & & & \star \end{bmatrix}\right\}$$

Specifically, assume $A = \sum_{k,\ell=1}^n a_{k\ell}E_{k\ell} \in \mathbf{G}_\alpha$ where $E_{k\ell}$ is an elementary matrix (as in Example 3.16) and $a_{k\ell} \in \mathbb{K}$ is a scalar, then we have $a_{k\ell} = 0$ unless $k = \ell$ or $(k,\ell) \in \{(i,j),(n+1-i,n+1-j)\}$. Assume now that $i < j$. Let I_n be the $n \times n$ identity matrix then one can easily check that the morphism φ_α is an isomorphism given by

$$\varphi_\alpha \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix} = I_n + cE_{ij} \qquad \varphi_\alpha \begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix} = I_n + cE_{n+1-i,n+1-j},$$

for all $c \in \mathbb{K}^+$. If $i > j$ then φ_α is obtained from the above description by exchanging the roles of the upper and lower uni-triangular matrices in $\mathrm{SL}_2(\mathbb{K})$.

Under the assumption that $i > j$ we have the root subgroup $\mathbf{X}_\alpha$ is contained in our fixed Borel subgroup $\mathbf{B}_0$ (consisting of upper triangular matrices). In particular, we obtain that the system of positive roots $\Phi^+(\mathbf{T}_0)$ determined by $\mathbf{B}_0$ is given by

$$\Phi^+(\mathbf{T}_0) = \{e_i - e_j \mid 1 \leqslant i < j \leqslant n+1\}.$$

For each $1 \leqslant k \leqslant n$ let us denote by $\check{e}_i \in \check{X}(\mathbf{T}_0)$ the homomorphism given by

$$\check{e}_k(\lambda) = \mathrm{diag}(1, \dots, 1, \lambda, 1, \dots, 1),$$

where $\lambda \in \mathbb{K}^\times$ is in the kth position. Using the description of the isomorphism φ_α given above we see that the coroot corresponding to α is given by

$$\check{\alpha}(\lambda) = \varphi_\alpha \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix} = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & \lambda & & & \\ & & & \ddots & & \\ & & & & \lambda^{-1} & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix} = (\check{e}_i - \check{e}_j)(\lambda)$$

for all $\lambda \in \mathbb{K}^\times$. With this description one easily confirms that the root system of $\mathrm{GL}_n(\mathbb{K})$ is isomorphic to the root datum described in Example 2.38.

With Theorem 3.23 in hand it seems sensible to ask if some of the group theoretic definitions made at the beginning of the section can be recast in terms of root data. The following deals with some of these questions.

Lemma 3.26. *Assume $\mathbf{G}$ is a connected reductive algebraic group with root datum Ψ then the following hold.*

(i) *$\mathbf{G}$ is semisimple if and only if $\mathbb{R}X$ is the $\mathbb{R}$-span of Φ.*

(ii) *$\mathbf{G}$ is simple if and only if Φ is indecomposable.*

Conversely, we may also make definitions using the root datum. For example, recall that if Ψ is the root datum of $\mathbf{G}$ then one has the weight lattice Λ of the root datum defined by the root system. Assume $\mathbf{G}$ is semisimple then we have a sequence of inclusions $\mathbb{Z}\Phi \subseteq X \subseteq \Lambda$.

Definition 3.27. We say a connected semisimple algebraic group $\mathbf{G}$ is *simply connected* if $X = \Lambda$ and *adjoint* if $X = \mathbb{Z}\Phi$.

Remark 3.28. Equivalently we have $\mathbf{G}$ is simply connected if $\check{X} = \mathbb{Z}\check{\Phi}$ and adjoint if $\check{X} = \check{\Lambda}$.

Exercise 3.29. Show that for each root system Φ there is a unique simply connected group and adjoint group up to isomorphism.

Exercise 3.30. Let us assume that $p = \mathrm{char}(\mathbb{K})$ is odd then for $n > 0$ we define two invertible matrices

$$Q_n = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ \vdots & & \iddots & 0 \\ 0 & \iddots & & \vdots \\ 1 & 0 & \cdots & 0 \end{bmatrix} \in M_n(\mathbb{K}) \qquad\qquad \Omega_n = \begin{bmatrix} 0 & Q_n \\ -Q_n & 0 \end{bmatrix} \in M_{2n}(\mathbb{K}).$$

With these matrices we may define the corresponding *orthogonal* and *symplectic groups* to be

$$\begin{aligned} \mathrm{O}_n(\mathbb{K}) &= \{A \in M_n(\mathbb{K}) \mid A^T Q_n A = Q_n\} \\ \mathrm{Sp}_{2n}(\mathbb{K}) &= \{A \in M_{2n}(\mathbb{K}) \mid A^T \Omega_n A = \Omega_n\}. \end{aligned}$$

We additionally define the *special orthgonal group* to be $\mathrm{SO}_n(\mathbb{K}) = \mathrm{O}_n(\mathbb{K}) \cap \mathrm{SL}_n(\mathbb{K})$. It is easy to verify that these are algebraic groups. The special linear, special orthogonal and symplectic groups are all connected reductive algebraic groups. Assume $\mathbf{G}$ is one of these groups then we have the corresponding Lie algebra is given by

$$\mathfrak{g} = \begin{cases} \{A \in M_n(\mathbb{K}) \mid \mathrm{tr}(A) = 0\} & \text{if } \mathbf{G} = \mathrm{SL}_n(\mathbb{K}) \\ \{A \in M_n(\mathbb{K}) \mid A^T Q_n + Q_n A = 0\} & \text{if } \mathbf{G} = \mathrm{SO}_n(\mathbb{K}) \\ \{A \in M_{2n}(\mathbb{K}) \mid A^T \Omega_n + \Omega_n A = 0\} & \text{if } \mathbf{G} = \mathrm{Sp}_{2n}(\mathbb{K}), \end{cases}$$

(see [Gec03, Theorem 1.5.13]).

Let $\bar{\mathbf{G}}$ be $\mathrm{GL}_n(\mathbb{K})$ (resp. $\mathrm{GL}_{2n}(\mathbb{K})$) if $\mathbf{G}$ is $\mathrm{SL}_n(\mathbb{K})$ or $\mathrm{SO}_n(\mathbb{K})$ (resp. $\mathrm{Sp}_{2n}(\mathbb{K})$) and let $\bar{\mathbf{T}}_0$ and $\bar{\mathbf{B}}_0$ be the maximal torus and Borel subgroup of $\bar{\mathbf{G}}$ consisting of diagonal matrices and upper triangular matrices respectively. Then $\mathbf{T}_0 := \bar{\mathbf{T}}_0 \cap \mathbf{G}$ and $\mathbf{B}_0 = \bar{\mathbf{B}}_0 \cap \mathbf{G}$ are respectively a maximal torus and Borel subgroup of $\mathbf{G}$ (see Theorem 1.7.4 and Proposition 3.4.6 of [Gec03]). Recall from [Hum75, §10.3 - Proposition] that, as in the case of the general linear group, we have $\mathrm{Ad}_x(y)$ is simply the matrix product xyx^{-1} for all $x \in \mathbf{G}$ and $y \in \mathfrak{g}$.

Using this, determine the root datum of $\mathbf{G}$. (Hint: use Lemmas 1.5.9, 1.5.10 and 1.5.11 of [Gec03].) Show that the root system of $\mathbf{G}$ is of type A_{n-1}, B_n, C_n or D_n if $\mathbf{G}$ is $\mathrm{SL}_n(\mathbb{K})$, $\mathrm{SO}_{2n+1}(\mathbb{K})$, $\mathrm{Sp}_{2n}(\mathbb{K})$ or $\mathrm{SO}_{2n}(\mathbb{K})$ respectively. Furthermore show that $\mathrm{SL}_n(\mathbb{K})$ and $\mathrm{Sp}_{2n}(\mathbb{K})$ are simply connected, $\mathrm{SO}_{2n+1}(\mathbb{K})$ is adjoint and $\mathrm{SO}_{2n}(\mathbb{K})$ is neither simply connected or adjoint. Finally, show that $\mathrm{SL}_2(\mathbb{K}) = \mathrm{Sp}_2(\mathbb{K})$. (Hint: use Theorem 3.23.)

∘ *Parabolic and Levi Subgroups*

Definition 3.31. Assume Φ is a root system with corresponding Weyl group W_Φ. We say $H \leqslant W_\Phi$ is a *parabolic subgroup* of W_Φ if there exists a simple system $\Delta \subset \Phi$ and a subset

$J \subseteq \Delta$ such that H is given by $W_J = \langle s_\alpha \mid \alpha \in J \rangle$.

Taking J to be Δ (resp. $\varnothing$) in the definition we obtain that W_Φ (resp. $\{1\}$) is a parabolic subgroup. If Δ is a fixed simple system and H is a parabolic subgroup of W_Φ then H is conjugate to W_J for some unique $J \subseteq \Delta$ (this follows from Theorem 2.12). After fixing a simple system, we sometimes call the subgroups W_J *standard parabolic subgroups*.

Example 3.32. Let $\Delta \subset \Phi^+ \subset \Phi$ be the root system of type A_{n-1} described in Example 2.31. Given $1 \leqslant m \leqslant n-1$ we take J to be $\Delta \setminus \{e_m - e_{m+1}\}$ then W_J is simply the Young subgroup of $\mathfrak{S}_n$ isomorphic to $\mathfrak{S}_m \times \mathfrak{S}_{n-m}$. Iterating this procedure we see that any standard parabolic subgroup of $\mathfrak{S}_n$ is simply a Young subgroup isomorphic to $\mathfrak{S}_{m_1} \times \cdots \times \mathfrak{S}_{m_r}$ for some non-zero $m_1, \dots, m_r \in \mathbb{N}$ satisfying $m_1 + \cdots + m_r = n$.

We would like to use the definition of parabolic subgroups for Weyl groups to define corresponding subgroups of connected reductive algebraic groups. To do this we will need the Bruhat decomposition, which gives a decomposition of $\mathbf{G}$ in terms of the Borel subgroup $\mathbf{B}_0$ and the Weyl group $W_{\mathbf{G}}(\mathbf{T}_0)$.

Theorem 3.33 (Bruhat Decomposition). *Assume $\mathbf{G}$ is a connected reductive algebraic group with maximal torus and Borel subgroup $\mathbf{T} \leqslant \mathbf{B}$. If $W = W_{\mathbf{G}}(\mathbf{T})$ is the corresponding Weyl group then we have*

$$\mathbf{G} = \bigsqcup_{w \in W} \mathbf{B}\dot{w}\mathbf{B}$$

where $\dot{w} \in N_{\mathbf{G}}(\mathbf{T})$ is a representative of $w \in W$ (note that this union is disjoint). Assume $\mathbb{S} \subset W$ is the set of simple reflections determined by $\mathbf{B}$. For any simple reflection $s \in \mathbb{S}$ and any $w \in W$ we have the multiplication of the corresponding double cosets is given by

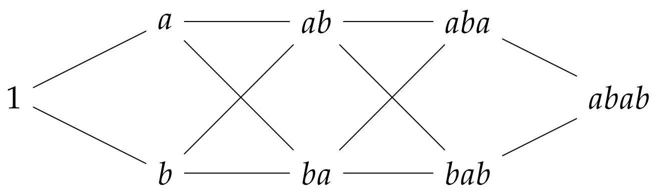
$$(\mathbf{B}\dot{s}\mathbf{B}) \cdot (\mathbf{B}\dot{w}\mathbf{B}) = \begin{cases} \mathbf{B}\dot{s}\dot{w}\mathbf{B} & \text{if } \ell(sw) = \ell(w) + 1 \\ \mathbf{B}\dot{w}\mathbf{B} \sqcup \mathbf{B}\dot{s}\dot{w}\mathbf{B} & \text{if } \ell(sw) = \ell(w) - 1 \end{cases}$$

A double coset $\mathbf{B}\dot{w}\mathbf{B}$ of $\mathbf{G}$ is usually called a *Bruhat cell* while its closure $\overline{\mathbf{B}\dot{w}\mathbf{B}}$ is called a *Schubert cell*. One can define an equivalence relation $\leqslant$ on $W_{\mathbf{G}}(\mathbf{T})$ by setting $w \leqslant v$ if and only if $\mathbf{B}\dot{w}\mathbf{B} \subseteq \overline{\mathbf{B}\dot{v}\mathbf{B}}$. The relation $\leqslant$ turns out to be equivalent to the *Bruhat ordering* on the Weyl group, which is defined as follows.

Definition 3.34. Assume $v, w \in W_\Phi$ and let $w = s_1 \cdots s_k$ be a reduced expression for w with each $s_i \in \mathbb{S}$. We write $v \leqslant w$ if $v = s_{i_1} \cdots s_{i_j}$ for some $1 \leqslant i_1 < i_2 < \cdots < i_j \leqslant k$. In other words, v occurs as a subexpression of w. We call $\leqslant$ the *Bruhat order* on W_Φ.

Exercise 3.35. Show that W_Φ has a unique maximal and unique minimal element with respect to the Bruhat ordering.

Example 3.36. Assume Φ is a root system of type $\mathsf{B}_2 = \mathsf{C}_2$ then $W_\Phi = \langle a, b \mid a^2 = b^2 = (ab)^4 = 1 \rangle$ is isomorphic to a dihedral group of order 8. The Bruhat order on W_Φ is then given by the following Hasse diagram.



Exercise 3.37. Describe the Bruhat ordering on W_Φ when Φ is of type A_3 or B_3.

We note that there are several equivalent ways to define the Bruhat order. With this digression over with we now consider parabolic subgroups of algebraic groups.

Definition 3.38. A closed subgroup $\mathbf{P}$ of a connected affine algebraic group $\mathbf{H}$ is called a *parabolic subgroup* if it contains a Borel subgroup of $\mathbf{H}$.

Exercise 3.39. Prove that every parabolic subgroup $\mathbf{P}$ of a connected affine algebraic group $\mathbf{H}$ is self normalising, i.e. $N_{\mathbf{H}}(\mathbf{P}) = \mathbf{P}$ (Hint: use Proposition 3.11).

We will assume that the maximal torus and Borel subgroup used in Theorem 3.33 are our fixed maximal torus and Borel subgroup $\mathbf{T}_0 \leqslant \mathbf{B}_0$ (in particular $W = W_{\mathbf{G}}(\mathbf{T}_0)$). Given a subset $J \subseteq \Delta(\mathbf{T}_0)$ we define

$$\mathbf{P}_J = \bigsqcup_{w \in W_J} \mathbf{B}_0 \dot{w} \mathbf{B}_0,$$

to be the *standard parabolic subgroup* of $\mathbf{G}$ defined by J.

Exercise 3.40. Prove that $\mathbf{P}_J$ is a parabolic subgroup of $\mathbf{G}$.

Proposition 3.41. *Any parabolic subgroup of* $\mathbf{G}$ *is conjugate to a unique standard parabolic subgroup of* $\mathbf{G}$.

This proposition shows that, as in the case of Weyl groups, the standard parabolic subgroups of an algebraic group are exemplary of all parabolic subgroups. Also note that, a corollary of the proposition is that for any $I, J \subseteq \Delta(\mathbf{T}_0)$ we have $\mathbf{P}_I = \mathbf{P}_J$ implies $I = J$.

Example 3.42. Let $\mathbf{G} = \mathrm{GL}_n(\mathbb{K})$ and let $\mathbf{T}_0 \leqslant \mathbf{B}_0$ be the maximal torus and Borel subgroup defined in Example 3.12. Let J be the set of simple roots $\Delta(\mathbf{T}_0) \setminus \{e_m - e_{m+1}\}$ for some $1 \leqslant m \leqslant n-1$ then we have

$$\mathbf{P}_J = \left\{ \begin{bmatrix} A & \star \\ 0 & B \end{bmatrix} \middle| A \in \mathrm{GL}_m(\mathbb{K}), B \in \mathrm{GL}_{n-m}(\mathbb{K}) \right\}.$$

Iterating this procedure one easily shows that every standard parabolic subgroup of $\mathbf{G}$ is of the form

$$\mathbf{P}_J = \begin{bmatrix} \star & \cdots & \cdots & \star \\ & \star & & \vdots \\ & & \ddots & \vdots \\ & 0 & \star & \vdots \\ & & & \star \end{bmatrix}$$

Definition 3.43. An algebraic group $\mathbf{H}$ is said to have a *Levi decomposition* if there exists a closed subgroup $\mathbf{L} \leqslant \mathbf{H}$ such that $\mathbf{H} = \mathbf{L} \ltimes R_u(\mathbf{H})$. We call $\mathbf{L}$ a *Levi subgroup*, or *Levi complement*, of $\mathbf{H}$.

Not all affine algebraic groups admit a Levi decomposition but it is clear that a reductive algebraic group admits a Levi decomposition (because the unipotent radical is trivial). Having said this, parabolic subgroups always have a Levi decomposition. For standard parabolic subgroups a decomposition can be constructed in the following way. Given $J \subseteq \Delta(\mathbf{T}_0)$ let us denote by

$$\Phi_J = \Phi \cap \left(\sum_{\alpha \in J} \mathbb{Z}\alpha \right)$$

the root subsystem generated by J. Setting $\Phi_J^+ = \Phi^+ \cap \Phi_J$ we obtain a positive system of roots for Φ_J and we define

$$\mathbf{L}_J = \langle \mathbf{T}_0, \mathbf{X}_\alpha \mid \alpha \in \Phi_J \rangle \qquad\qquad \mathbf{U}_J = \langle \mathbf{X}_\alpha \mid \alpha \in \Phi^+ \setminus \Phi_J^+ \rangle.$$

Exercise 3.44. Prove that $\mathbf{L}_J$ and $\mathbf{U}_J$ are subgroups of $\mathbf{P}_J$.

Lemma 3.45. *For any $J \subseteq \Delta(\mathbf{T}_0)$ we have $R_u(\mathbf{P}_J) = \mathbf{U}_J$. Furthermore, $\mathbf{P}_J$ has a Levi decomposition given by $\mathbf{P}_J = \mathbf{L}_J \ltimes \mathbf{U}_J$ and $\mathbf{L}_J$ is a connected reductive algebraic group.*

We call $\mathbf{L}_J$ a *standard Levi subgroup* of $\mathbf{P}_J$. Sometimes we will say that a Levi subgroup $\mathbf{L}$ of a parabolic subgroup of $\mathbf{G}$ is a *Levi subgroup of* $\mathbf{G}$. In this situation we say $\mathbf{L}_J$ is a *standard Levi subgroup* of $\mathbf{G}$.

Example 3.46. Returning to Example 3.42 we have

$$\mathbf{L}_J = \left\{ \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \middle| A \in \mathrm{GL}_m(\mathbb{K}), B \in \mathrm{GL}_{n-m}(\mathbb{K}) \right\} \qquad\qquad \mathbf{U}_J = \left\{ \begin{bmatrix} I_m & \star \\ 0 & I_{n-m} \end{bmatrix} \right\}.$$

In the general case of a standard parabolic subgroup of $\mathrm{GL}_n(\mathbb{K})$ one obtains the Levi complement and unipotent radical to be

$$\mathbf{L}_J = \begin{bmatrix} \star & & & & \\ & \star & & 0 & \\ & & \ddots & & \\ & 0 & & \star & \\ & & & & \star \end{bmatrix} \qquad \mathbf{U}_J = \begin{bmatrix} I & & & & \\ & I & & \star & \\ & & \ddots & & \\ & 0 & & I & \\ & & & & I \end{bmatrix}$$

respectively.

Although this is one Levi decomposition of the standard parabolic subgroup $\mathbf{P}_J$, it is not necessarily the only one. However, the Levi complement is uniquely determined by the condition that it contains the maximal torus $\mathbf{T}_0$. In general we have the following general result.

Lemma 3.47. *Assume $\mathbf{P} \leqslant \mathbf{G}$ is a parabolic subgroup and let $\mathbf{T} \leqslant \mathbf{P}$ be a maximal torus of $\mathbf{G}$ then there exists a unique Levi subgroup $\mathbf{L} \leqslant \mathbf{P}$ which contains $\mathbf{T}$. Furthermore, any two Levi subgroups of $\mathbf{P}$ are conjugate by a unique element of $R_u(\mathbf{P})$.*

We end our discussion of parabolic and Levi subgroups by noting that the analogue of Proposition 3.41 for Levi subgroups does not hold. In particular, there may exist subsets $I, J \subseteq \Delta(\mathbf{T}_0)$ such that $\mathbf{L}_I$ and $\mathbf{L}_J$ are conjugate in $\mathbf{G}$ but $I \neq J$.

Exercise 3.48. Construct such an example. (Hint: use conjugation by elements of the Weyl group $W_{\mathbf{G}}(\mathbf{T}_0)$.)

◦ *Conjugacy Classes and Centralisers*

When trying to understand a group one of the first things to be done is to try and understand its conjugacy classes and centralisers of elements. The conjugacy classes of affine algebraic groups have been well studied. In particular, the theory of unipotent conjugacy classes is a rich area with many links to geometry. Here, we simply give some of the ideas towards understanding the conjugacy classes of a connected reductive algebraic group.

Definition 3.49. We say $x \in \mathrm{GL}_n(\mathbb{K})$ is *semisimple* if it is a diagonalisable matrix and *unipotent* if all its eigenvalues are 1.

Theorem 3.50 (Jordan decomposition). *Assume $\mathbf{H}$ is an affine algebraic group and $\rho : \mathbf{H} \to \mathrm{GL}_n(\mathbb{K})$ is an embedding. For any element $g \in \mathbf{H}$ there exists $s, u \in \mathbf{H}$ such that $g = su = us$, where $\rho(s)$ is semisimple and $\rho(u)$ is unipotent. Furthermore, the decomposition $g = su = us$ is independent of the choice of embedding ρ.*

Exercise 3.51. Show that any conjugate of a semisimple (resp. unipotent) element of an affine algebraic group $\mathbf{H}$ is again semisimple (resp. unipotent).

For any element $x \in \mathbf{G}$ we will write $(x) = (x)_{\mathbf{G}}$ for the conjugacy class of $\mathbf{G}$ containing x. We say a conjugacy class is semisimple (resp. unipotent) if it consists entirely of semisimple (resp. unipotent) elements. Furthermore we say a conjugacy class is *mixed* if it consists of elements that are neither semisimple nor unipotent. In light of the Jordan decomposition we will focus our attention on semisimple and unipotent conjugacy classes. The case of semisimple elements is dealt with by the following result.

Proposition 3.52. *Assume $\mathbf{G}$ is a connected reductive algebraic group then the map $t \mapsto (t)$ induces a bijection*

$$\mathbf{T}_0/W_{\mathbf{G}}(\mathbf{T}_0) \to \{\textit{semisimple conjugacy classes of } \mathbf{G}\},$$

where $\mathbf{T}_0/W_{\mathbf{G}}(\mathbf{T}_0)$ denote the $W_{\mathbf{G}}(\mathbf{T}_0)$-orbits of $\mathbf{T}_0$ under the action $t \mapsto t^w$.

In particular, this result says that a representative for every semisimple conjugacy class may be found in our fixed maximal torus $\mathbf{T}_0$.

Example 3.53. Consider the case where $\mathbf{G} = \mathrm{GL}_3(\mathbb{K})$. Firstly, every semisimple element of $\mathbf{G}$ is conjugate to a diagonal matrix contained in $\mathbf{T}_0$. Using Proposition 3.52 we now need only determine the orbits of $W_{\mathbf{G}}(\mathbf{T}_0)$-acting on $\mathbf{T}_0$. One easily checks that the orbit of an element $(t_1, t_2, t_3) = \mathrm{diag}(t_1, t_2, t_3) \in \mathbf{T}_0$ is given by

- $\{(t_1, t_2, t_3), (t_1, t_3, t_2), (t_2, t_1, t_3), (t_2, t_3, t_1), (t_3, t_2, t_1), (t_3, t_1, t_2)\}$ if t_1, t_2 and t_3 are all distinct.
- $\{(t_1, t_1, t_3), (t_1, t_3, t_1), (t_3, t_1, t_1)\}$ if $t_1 = t_2$ but $t_1 \neq t_3$.
- $\{(t_1, t_1, t_1)\}$ if $t_1 = t_2 = t_3$.

Observe that this generalises the usual fact that, in $\mathrm{GL}_n(\mathbb{K})$, one has two semisimple matrices are diagonalisable if and only if they have the same eigenvalues.

As every semisimple element of $\mathbf{G}$ is conjugate to some element of $\mathbf{T}_0$ one may hope that the centraliser of an element in $\mathbf{T}_0$ can be described in terms of the roots of $\mathbf{G}$ relative to $\mathbf{T}_0$.

Lemma 3.54 (Steinberg). *Assume $s \in \mathbf{T}_0$ is semisimple then we have*

$$C_{\mathbf{G}}(s) = \langle \mathbf{T}_0, \mathbf{X}_\alpha, \dot{w} \mid \alpha \in \Phi(\mathbf{T}_0), \alpha(s) = 1 \textit{ and } s^w = s \rangle,$$
$$C_{\mathbf{G}}(s)^\circ = \langle \mathbf{T}_0, \mathbf{X}_\alpha \mid \alpha \in \Phi(\mathbf{T}_0) \textit{ and } \alpha(s) = 1 \rangle.$$

In particular $C_{\mathbf{G}}(s)$ is reductive and the root system of its connected component (relative to $\mathbf{T}_0$) is given by $\Phi(s) = \{\alpha \in \Phi(\mathbf{T}_0) \mid \alpha(s) = 1\}$.

Exercise 3.55. Assume $\mathbf{G} = \mathrm{GL}_3(\mathbb{K})$ then determine the centraliser of the semisimple element $\mathrm{diag}(t_1, t_2, t_3) \in \mathbf{T}_0$ (for all possible choices of t_i). Show that the centraliser is always connected and is a standard Levi subgroup of $\mathbf{G}$.

Exercise 3.56. Construct a semisimple element of $\mathbf{G} = \mathrm{Sp}_4(\mathbb{K})$ whose centraliser is connected but is not a Levi subgroup of $\mathbf{G}$.

As the previous exercise illustrates, although $\Phi(s)$ is a root system in $\Phi(\mathbf{T}_0)$ it is not necessarily the root system of a Levi subgroup. However, one may easily deduce from Lemma 3.54, that (up to isomorphism) there exist only finitely many different centralisers of semisimple elements in $\mathbf{G}$. This proves promising when trying to construct a list of all conjugacy classes of $\mathbf{G}$. In particular, the Jordan decomposition shows us that any conjugacy class of $\mathbf{G}$ is a product $(s)_{\mathbf{G}} \cdot (u)_{C_{\mathbf{G}}(s)}$ where $(u)_{C_{\mathbf{G}}(s)}$ is a unipotent conjugacy class of $C_{\mathbf{G}}(s)$. This means we may inductively compute the conjugacy classes and centralisers in $\mathbf{G}$. In fact we may restrict ourselves to connected reductive algebraic groups using the following.

Lemma 3.57. *Assume $s \in \mathbf{G}$ is a semisimple element then any unipotent element $u \in C_{\mathbf{G}}(s)$ is contained in $C_{\mathbf{G}}(s)^{\circ}$.*

For a connected reductive algebraic group $\mathbf{G}$ it was shown by Lusztig (and also Richardson when p is *good* for $\mathbf{G}$) that there are only finitely many unipotent conjugacy classes of $\mathbf{G}$. Note also that a unipotent element of $\mathbf{G}$ must be contained in the derived subgroup (see Proposition 3.9). In fact, the classification of unipotent conjugacy classes can be reduced to the case of simple algebraic groups.

Lemma 3.58. *Assume $\mathbf{G}$ is a connected reductive algebraic group $\mathbf{G}$ with root system $\Phi(\mathbf{T}_0)$ with indecomposable subsystems $\Phi_i \subseteq \Phi(\mathbf{T}_0)$ such that $\Phi(\mathbf{T}_0) = \Phi_1 \sqcup \cdots \sqcup \Phi_r$. Let $\mathbf{G}_i = \langle X_\alpha \mid \alpha \in \Phi_i \rangle$ then $\mathbf{G}_i \leqslant \mathbf{G}$ is a simple algebraic group and the following hold.*

(i) $[\mathbf{G}, \mathbf{G}] = \mathbf{G}_1 \cdots \mathbf{G}_r$

(ii) $[\mathbf{G}_i, \mathbf{G}_j] = 1$ *for any* $i \neq j$

(iii) $\mathbf{G}_i \cap \mathbf{G}_1 \cdots \mathbf{G}_{i-1}\mathbf{G}_{i+1} \cdots \mathbf{G}_r$ *is finite for all* $1 \leqslant i \leqslant r$.

Let $u \in \mathbf{G}$ be a unipotent element then there exist unique unipotent elements $u_i \in \mathbf{G}_i$ such that $u = u_1 \cdots u_r$ and $u_i u_j = u_j u_i$ for all $i \neq j$. In particular, we have the conjugacy class $(u)_{\mathbf{G}}$ is the product $(u_1)_{\mathbf{G}_1} \cdots (u_r)_{\mathbf{G}_r}$.

Example 3.59. Assume $\mathbf{G} = \mathrm{GL}_3(\mathbb{K})$ then the unipotent classes of $\mathbf{G}$ are parameterised by the Jordan blocks of the matrix. In particular, the unipotent conjugacy classes of $\mathbf{G}$ are in bijection with the partitions of 3. Representatives are given by

$$u_{(1^3)} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \qquad u_{(2,1)} = \begin{bmatrix} 1 & 1 & \\ & 1 & \\ & & 1 \end{bmatrix} \qquad u_{(3)} = \begin{bmatrix} 1 & 1 & \\ & 1 & 1 \\ & & 1 \end{bmatrix}.$$

This, together with Example 3.53, gives all the semisimple and unipotent classes of $\mathbf{G}$.

Exercise 3.60. Give a complete list of class representatives for $\mathrm{GL}_3(\mathbb{K})$ by writing down representatives for the mixed classes.

∘ *Duality*

In Exercise 2.34 we observed that root data admit a duality given by $(X, \Phi, \check{X}, \check{\Phi}) \mapsto (\check{X}, \check{\Phi}, X, \Phi)$. By Theorem 3.23 both these root data correspond to connected reductive algebraic groups (up to isomorphism). In particular, this duality may be translated into a duality between connected reductive algebraic groups.

Definition 3.61. We define a *dual group* $\mathbf{G}^\star$ of $\mathbf{G}$ to be a connected reductive algebraic group such that for some maximal torus $\mathbf{T}_0^\star$ of $\mathbf{G}^\star$ we have an isomorphism of root data

$$\varphi : \Psi(\mathbf{T}_0) \to (\check{X}(\mathbf{T}_0), \check{\Phi}(\mathbf{T}_0), X(\mathbf{T}_0), \Phi(\mathbf{T}_0)).$$

In particular, there exists an isomorphism $\varphi : X(\mathbf{T}_0) \to \check{X}(\mathbf{T}_0^\star)$ of abelian groups. We say the isomorphism φ *defines the duality*.

We will see later that this notion of duality plays an important role in the representation theory of finite reductive groups. Now, let $\mathbf{G}^\star$ be a dual group of $\mathbf{G}$ and let $\varphi : X(\mathbf{T}_0) \to \check{X}(\mathbf{T}_0^\star)$ be an isomorphism inducing the duality. Similar to the discussion proceeding Definition 2.39 such an isomorphism gives rise to an isomorphism

$$\mathrm{Hom}(X(\mathbf{T}_0), \mathbb{Z}) \to \mathrm{Hom}(\check{X}(\mathbf{T}_0^\star), \mathbb{Z}),$$

hence to an isomorphism $\check{X}(\mathbf{T}_0) \to X(\mathbf{T}_0^\star)$. The two isomorphisms determine one another.

From the definition of an isomorphism between root data (see Definition 2.42) we are ensured that the following diagram of bijections is commutative.

$$
\begin{tikzcd}
\Phi(\mathbf{T}_0) \arrow[r, "\varphi"] \arrow[d, "\smile"'] & \check{\Phi}(\mathbf{T}_0^\star) \arrow[d, "\smile"] \\
\check{\Phi}(\mathbf{T}_0) \arrow[r, "\check{\varphi}"'] & \Phi(\mathbf{T}_0^\star)
\end{tikzcd}
$$

Our choice of Borel subgroup $\mathbf{B}_0$ determines a set of simple roots $\Delta(\mathbf{T}_0) \subset \Phi(\mathbf{T}_0)$. Taking $\Delta(\mathbf{T}_0^\star)$ to be the image of $\Delta(\mathbf{T}_0)$ under the above square gives a simple system for $\Phi(\mathbf{T}_0^\star)$ and in turn determines a Borel subgroup $\mathbf{B}_0^\star$ of $\mathbf{G}^\star$. Additionally, we have $\mathbb{S}^\star = \{s_\alpha \mid \alpha \in \Delta(\mathbf{T}_0^\star)\}$ is a set of simple reflections for the Weyl group of $\mathbf{G}^\star$.

Remark 3.62. In this situation we sometimes say the triples $(\mathbf{T}_0, \mathbf{B}_0, \mathbf{G})$ and $(\mathbf{T}_0^\star, \mathbf{B}_0^\star, \mathbf{G}^\star)$ are in *duality*.

Assume $\mu : X(\mathbf{T}_0) \to X(\mathbf{T}_0)$ is an endomorphism then define $\check{\mu}^\star : \check{X}(\mathbf{T}_0^\star) \to \check{X}(\mathbf{T}_0^\star)$ to be the endomorphism $\varphi \circ \mu \circ \varphi^{-1}$. Taking the dual of this endomorphism we obtain a resulting endomorphism $\mu^\star : X(\mathbf{T}_0^\star) \to X(\mathbf{T}_0^\star)$. This gives us maps

$$
\begin{aligned}
\mathrm{End}(X(\mathbf{T}_0)) &\to \mathrm{End}(X(\mathbf{T}_0^\star)) \\
\mathrm{End}(X(\mathbf{T}_0^\star)) &\to \mathrm{End}(X(\mathbf{T}_0))
\end{aligned}
$$

each denoted by $\mu \mapsto \mu^\star$ and satisfying $\mu^{\star\star} = \mu$. Recall that any element of the Weyl group $w \in W_{\mathbf{G}}(\mathbf{T}_0)$ may be viewed as an endomorphism of $X(\mathbf{T}_0)$, then we have the following result.

Lemma 3.63. *The map $w \mapsto w^\star$ is an anti-isomorphism $W_{\mathbf{G}}(\mathbf{T}_0) \to W_{\mathbf{G}^\star}(\mathbf{T}_0^\star)$ such that for each $\alpha \in \Phi(\mathbf{T}_0)$ we have $s_\alpha^\star = s_{\widecheck{\varphi(\alpha)}}$, (in other words this map restricts to a bijection between $\mathbb{S}$ and $\mathbb{S}^\star$).*

4. **Finite Reductive Groups**

◦ *Generalised Frobenius Endomorphisms*

Assume $q = p^a$ is a power of p for some $a \in \mathbb{N}$ then, as a field, $\mathbb{K}$ admits an automorphism σ given by $\sigma(x) = x^q$. We denote by $\mathbb{F}_q$ the set of fixed points $\mathbb{K}^\sigma$ under σ, which is simply the finite field of cardinality q. Note that σ is *not* an automorphism of the algebraic group $\mathbb{K}^+$ but simply a bijective homomorphism (the inverse σ^{-1} is not a regular map).

We would now like to extend this construction to affine algebraic groups. Firstly, we define a bijective regular map $F_q : \mathrm{Mat}_n(\mathbb{K}) \to \mathrm{Mat}_n(\mathbb{K})$ by setting

$$F_q(x_{ij}) = (x_{ij}^q).$$

We call F_q the *standard Frobenius endomorphism* of $\mathrm{Mat}_n(\mathbb{K})$. Note that the set of fixed points $\mathrm{Mat}_n(\mathbb{K})^{F_q}$ under F_q is simply the set of all matrices $\mathrm{Mat}_n(q) := \mathrm{Mat}_n(\mathbb{F}_q)$ with coefficients in the finite field $\mathbb{F}_q$.

Assume now that $\mathbf{H}$ is an affine algebraic group and $\phi : \mathbf{H} \hookrightarrow \mathrm{Mat}_n(\mathbb{K})$ is a closed embedding (tacitly assumed to be a group homomorphism). If F_q stabilises $\phi(\mathbf{H})$, i.e. $F_q(\phi(\mathbf{H})) = \phi(\mathbf{H})$, then we call $F_{q,\phi} = \phi^{-1} \circ F_q \circ \phi$ a *standard Frobenius endomorphism* of $\mathbf{H}$. In other words, identifying $\mathbf{H}$ with $\phi(\mathbf{H})$, we may think of $F_{q,\phi}$ as the restriction of F_q to $\mathbf{H}$. The fixed point group

$$\mathbf{H}^{F_{q,\phi}} = \{h \in \mathbf{H} \mid F_{q,\phi}(h) = h\}$$

is a finite subgroup of $\mathbf{H}$ (note that it is isomorphic to the fixed point group $\phi(\mathbf{H})^{F_q} \subseteq \mathrm{Mat}_n(q)$). For example, taking $\mathbf{H}$ to be $\mathrm{GL}_n(\mathbb{K})$ and $\phi : \mathbf{H} \hookrightarrow \mathrm{Mat}_n(\mathbb{K})$ to be the natural inclusion we have the fixed point group $\mathrm{GL}_n(\mathbb{K})^{F_{q,\phi}}$ is the finite general linear group $\mathrm{GL}_n(q)$.

Exercise 4.1. Show that for any integer $k > 0$ and any standard Frobenius endomorphism $F_{q,\phi} : \mathbf{H} \to \mathbf{H}$ we have $F_{q,\phi}^k = F_{q,\phi} \circ \cdots \circ F_{q,\phi} = F_{q^k,\phi}$.

Remark 4.2. Our notation here may seem a bit clumsy. However, we wish to emphasise the fact that $F_{q,\phi}$ depends upon the embedding ϕ. This will pay off when we compare the idea of a standard Frobenius endomorphism and a Frobenius endomorphism.

Example 4.3. Take $\mathbf{H}$ to be $\mathrm{SL}_n(\mathbb{K})$, $\mathrm{O}_n(\mathbb{K})$ or $\mathrm{SO}_n(\mathbb{K})$ (see Exercise 3.30) and $\phi : \mathbf{H} \hookrightarrow \mathrm{Mat}_n(\mathbb{K})$ to be the natural inclusion map then F_q preserves $\phi(\mathbf{H})$ and we have the fixed point group $\mathbf{H}^{F_{q,\phi}}$ is the finite group $\mathrm{SL}_n(q)$, $\mathrm{O}_n(q)$ or $\mathrm{SO}_n(q)$ respectively. Similarly we may take $\mathbf{H} = \mathrm{Sp}_{2n}(\mathbb{K})$ and $\phi : \mathbf{H} \hookrightarrow \mathrm{GL}_{2n}(\mathbb{K})$ to be the natural inclusion map to obtain the finite symplectic group $\mathbf{H}^{F_{q,\phi}} = \mathrm{Sp}_{2n}(q)$.

Using the definition given above it is easy to produce standard Frobenius endomorphisms of affine algebraic groups. However, it would be desirable to have a definition which does not depend upon the closed embedding ϕ. Such a definition can be obtained by using the affine algebra.

Definition 4.4. Assume $\mathbf{H}$ is an affine algebraic group with affine algebra $A = \mathbb{K}[\mathbf{H}]$. Let us recall the notation of Proposition 3.2. We say a regular map $F : \mathbf{H} \to \mathbf{H}$ is a *Frobenius endomorphism* if there exists a power $q = p^a$, with $a \in \mathbb{N}$, such that

(i) F^* is injective and $F^*(A) = A^q$.

(ii) For each $f \in A$ there exists some $m \geqslant 1$ such that $(F^*)^m(f) = f^{q^m}$.

Similarly, we say $\mathbf{H}$ is *defined over* $\mathbb{F}_q$ (or admits an $\mathbb{F}_q$*-rational structure*) with corresponding Frobenius endomorphism F. Additionally, we say F is a *generalised Frobenius endomorphism* if F^m is a Frobenius endomorphism for some $m > 0$.

Example 4.5. Assume $\mathbf{H} \subseteq \mathrm{Mat}_n(\mathbb{K})$ is an affine algebraic group such that $F_q(\mathbf{H}) = \mathbf{H}$. We claim that $F_{q,\phi}$ is a Frobenius endomorphism, where $\phi : \mathbf{H} \hookrightarrow \mathrm{Mat}_n(\mathbb{K})$ is the natural inclusion map. Firstly we have the vanishing ideal $I(\mathbf{H})$ is an ideal of the polynomial ring $\mathbb{K}[X_{ij} \mid 1 \leqslant i,j \leqslant n]$. The condition that $\mathbf{H}$ is stable under F_q is equivalent to the condition that $I(\mathbf{H})$ is stable under F_q^*. In particular, F_q^* induces a $\mathbb{K}$-algebra homomorphism $F_{q,\phi}^*$ of the affine algebra $A = \mathbb{K}[\mathbf{H}]$ whose corresponding regular map is $F_{q,\phi}$.

The map $F_{q,\phi}$ is bijective hence $F_{q,\phi}^*$ is injective (see Proposition 3.2). As $F_q^*(X_{ij}) = X_{ij}^q$ for all $1 \leqslant i,j \leqslant n$ we must have $F_{q,\phi}^*(A) = A^q$, which shows that (i) holds. To see that (ii) holds we recall $\mathbb{K}$ can be expressed as the infinite union $\cup_{m=1}^{\infty} \mathbb{F}_{p^m}$, in particular any element of $\mathbb{K}$ lies in a finite subfield. Consequently, for any $f \in A$ there exists $m \geqslant 1$ such that all coefficients of f lie in $\mathbb{F}_{q^m}$ hence $(F_{q,\phi}^*)^m(f) = f^{q^m}$ as required.

Exercise 4.6. Assume $\mathbf{H}$ is an affine algebraic group defined over $\mathbb{F}_q$ with corresponding Frobenius endomorphism $F : \mathbf{H} \to \mathbf{H}$. Let $\varphi : \mathbf{H} \to \mathbf{H}$ be a regular map such that $\varphi^k = \mathrm{id}$ for some $k \geqslant 1$ and $\varphi \circ F = F \circ \varphi$. Show that $F' = F \circ \varphi$ is also a Frobenius endomorphism of $\mathbf{H}$ admitting an $\mathbb{F}_q$-rational structure (Hint: use Proposition 3.2 and work in the affine algebra).

Using this exercise we may now introduce another Frobenius endomorphism on the general linear group $\mathrm{GL}_n(\mathbb{K})$.

Example 4.7. Assume $\mathbf{H} = \mathrm{GL}_n(\mathbb{K})$ and let $\phi : \mathbf{H} \hookrightarrow \mathrm{Mat}_n(\mathbb{K})$ be the natural inclusion morphism then we obtain the standard Frobenius endomorphism $F_{q,\phi}$ on $\mathbf{H}$. Let $\tau : \mathbf{H} \to \mathbf{H}$ be the inverse-transpose automorphism given by $\tau(A) = A^{-T}$. Setting $F = F_{q,\phi} \circ \tau$ we have $F^2 = F_{q^2,\phi}$ because $F_{q,\phi} \circ \tau = \tau \circ F_{q,\phi}$ and $\tau^2 = 1$, hence F is certainly a generalised Frobenius endomorphism. However, by the previous exercise, we have F is a Frobenius endomorphism admitting an $\mathbb{F}_q$-rational structure of $\mathbf{H}$. The fixed point group $\mathbf{H}^F \leqslant \mathbf{H}^{F^2} = \mathrm{GL}_n(q^2)$ is the finite unitary group $\mathrm{U}_n(q)$.

In this example we can see that the Frobenius endomorphism F of $\mathbf{H} = \mathrm{GL}_n(\mathbb{K})$ is not the standard Frobenius endomorphism $F_{q,\phi}$ where $\phi : \mathbf{H} \hookrightarrow \mathrm{Mat}_n(\mathbb{K})$ is the natural inclusion. However, somewhat confusingly, there exists a closed embedding $\sigma : \mathbf{H} \hookrightarrow \mathrm{Mat}_m(\mathbb{K})$ such that F is the standard Frobenius endomorphism $F_{q,\sigma}$. Note that it is not necessarily the case that $m = n$. In general, we have the following result which shows that any Frobenius endomorphism is a standard Frobenius endomorphism (as defined above).

Proposition 4.8. *Assume* $\mathbf{H}$ *is an affine algebraic group defined over* $\mathbb{F}_q$ *with corresponding Frobenius endomorphism* $F : \mathbf{H} \to \mathbf{H}$*. Then there exists a closed embedding* $\sigma : \mathbf{H} \to \mathrm{Mat}_m(\mathbb{K})$*, for some* $m > 0$*, such that the following diagram is commutative*

$$\begin{tikzcd} \mathbf{H} \arrow[r, "\sigma"] \arrow[d, "F"'] & \mathrm{Mat}_m(\mathbb{K}) \arrow[d, "F_q"] \\ \mathbf{H} \arrow[r, "\sigma"'] & \mathrm{Mat}_m(\mathbb{K}) \end{tikzcd}$$

In other words, F *is the standard Frobenius endomorphism* $F_{q,\sigma}$*.*

We will not give details here but we merely state that not every generalised Frobenius endomorphism is a Frobenius endomorphism.

Definition 4.9. We say a finite group G is a *finite reductive group* if there exists a connected reductive algebraic group $\mathbf{G}$ and a generalised Frobenius endomorphism F such that $G = \mathbf{G}^F$.

Remark 4.10. Note that the finite reductive group G in the above definition is not necessarily uniquely determined by the pair $(\mathbf{G}, F)$. In particular, there may exist a different pair $(\mathbf{G}', F')$ such that $G = \mathbf{G}'^{F'}$.

When studying finite reductive groups our main philosophy is to obtain information about G from the ambient algebraic group $\mathbf{G}$ through the generalised Frobenius endomorphism F. The following innocuous looking result is the key to this philosophy. We will see its full power in the following section.

Theorem 4.11 (Lang–Steinberg). *Assume* $\mathbf{H}$ *is a* connected *affine algebraic group and* $F : \mathbf{H} \to \mathbf{H}$ *is a generalised Frobenius endomorphism. Then the morphism* $\mathscr{L} : \mathbf{H} \to \mathbf{H}$ *defined by* $\mathscr{L}(g) = g^{-1}F(g)$ *is surjective.*

◦ *Parameterising Orbits*

When trying to determine the character table of a finite group, one of the first things to be done is to determine the conjugacy classes. We would like to develop a systematic way to describe the conjugacy classes of a finite reductive group. We will see that this, and much more, can be achieved using the Lang–Steinberg theorem.

Let us assume that $\mathbf{G}$ is a connected reductive algebraic group and $F : \mathbf{G} \to \mathbf{G}$ is a generalised Frobenius endomorphism. Furthermore, let us assume that $\mathbf{G}$ acts transitively on a non-empty set $\mathfrak{X}$ under the action $\cdot : \mathbf{G} \times \mathfrak{X} \to \mathfrak{X}$ and that there exists a map $F' : \mathfrak{X} \to \mathfrak{X}$ such that:

(a) $F'(g \cdot x) = F(g) \cdot F'(x)$ for all $g \in \mathbf{G}$ and $x \in \mathfrak{X}$.

(b) the stabiliser of any point $\mathrm{Stab}_{\mathbf{G}}(x)$ is a closed subgroup of $\mathbf{G}$.

Denote by $\mathfrak{X}^{F'} = \{x \in \mathfrak{X} \mid F'(x) = x\}$ the set of F'-fixed points (sometimes called rational points) then the group $G = \mathbf{G}^F$ acts naturally on $\mathfrak{X}^{F'}$ but this action is not, in general, transitive. One of the first, important, consequences of the Lang–Steinberg theorem is the following.

Lemma 4.12. *The set of fixed points $\mathfrak{X}^{F'}$ is non-empty.*

We would now like to consider how to obtain a natural method for parameterising the orbits of G acting on $\mathfrak{X}^{F'}$, which we denote by $\mathfrak{X}^{F'}/G$. We will call the orbits in $\mathfrak{X}^{F'}/G$ the *rational orbits* of $\mathfrak{X}^{F'}$. Let us fix an element $x_0 \in \mathfrak{X}^{F'}$ (which exists by Lemma 4.12) and assume for some $g \in \mathbf{G}$ that $g \cdot x_0 \in \mathfrak{X}^{F'}$ then

$$F'(g \cdot x_0) = g \cdot x_0 \Rightarrow (g^{-1}F(g)) \cdot x_0 = x_0 \Rightarrow \mathscr{L}(g) = g^{-1}F(g) \in \mathrm{Stab}_{\mathbf{G}}(x_0).$$

Let us write $A_{\mathbf{G}}(x_0)$ for the component group $\mathrm{Stab}_{\mathbf{G}}(x_0)/\mathrm{Stab}_{\mathbf{G}}(x_0)^\circ$ then $\mathscr{L}(g)$ naturally determines an element $\overline{\mathscr{L}(g)}$ in $A_{\mathbf{G}}(x_0)$.

Definition 4.13. Let H be a group and $\varphi : H \to H$ a homomorphism. We say two elements $x, y \in H$ are *φ-conjugate* if there exists $h \in H$ such that $x = h^{-1}y\varphi(h)$. This forms an equivalence relation on H and we call the equivalence classes the *φ-conjugacy classes* of H. We denote the set of all φ-conjugacy classes by $H^1(\varphi, H)$.

Theorem 4.14. *The map $g \cdot x_0 \mapsto \overline{\mathscr{L}(g)}$ induces a bijection*

$$\mathfrak{X}^{F'}/G \to H^1(F', A_{\mathbf{G}}(x_0)).$$

Remark 4.15. Note that this parameterisation heavily depends upon the choice of element $x_0 \in \mathfrak{X}^{F'}$. Changing the element x_0 may drastically change the action of F' on $A_{\mathbf{G}}(x_0)$, hence may fundamentally change the description of the parameterising set $H^1(F', A_{\mathbf{G}}(x_0))$.

Example 4.16. Let $\mathfrak{B} = \{\mathbf{B} \leqslant \mathbf{G} \mid \mathbf{B}$ is a Borel subgroup of $\mathbf{G}\}$ then according to Proposition 3.11 we have $\mathbf{G}$ acts transitively on $\mathfrak{B} \neq \varnothing$ by conjugation. For any $\mathbf{B} \in \mathfrak{B}$ we have the stabiliser $\mathrm{Stab}_{\mathbf{G}}(\mathbf{B})$ is simply the normaliser $N_{\mathbf{G}}(\mathbf{B})$, hence the action satisfies (b) above. We have a map $F' : \mathfrak{B} \to \mathfrak{B}$ given by $\mathbf{B} \mapsto F(\mathbf{B})$ and it is easy to check that F' satisfies (a). According to Lemma 4.12 we may assume that our chosen Borel subgroup $\mathbf{B}_0$ is fixed by F', i.e. $F(\mathbf{B}_0) = \mathbf{B}_0$. Using the fact that $N_{\mathbf{G}}(\mathbf{B}_0) = \mathbf{B}_0$ is connected (see Proposition 3.11), we have by Theorem 4.14 that $\mathfrak{B}^{F'}$ form a single rational orbit. In particular, any two F-stable Borel subgroups of $\mathbf{G}$ are conjugate by an element of G.

Example 4.17. Let $\mathfrak{T} = \{\mathbf{T} \leqslant \mathbf{G} \mid \mathbf{T} \text{ is a maximal torus of } \mathbf{G}\}$ then according to Proposition 3.11 $\mathbf{G}$ acts transitively on $\mathfrak{T}$ by conjugation. Again taking $F' : \mathfrak{T} \to \mathfrak{T}$ to be the map $\mathbf{T} \mapsto F(\mathbf{T})$ we have the conjugation action and F' satisfy (a) and (b) above. For any $\mathbf{T} \in \mathfrak{T}$ we again have the stabiliser $\mathrm{Stab}_{\mathbf{G}}(\mathbf{T})$ is simply the normaliser $N_{\mathbf{G}}(\mathbf{T})$ whose connected component is $C_{\mathbf{G}}(\mathbf{T}) = \mathbf{T}$. Let us assume that our chosen maximal torus $\mathbf{T}_0$ is F-stable. The component group $A_{\mathbf{G}}(\mathbf{T}_0)$ is simply the Weyl group $W_{\mathbf{G}}(\mathbf{T}_0)$ and Theorem 4.14 says that there is a bijection between the G-conjugacy classes of F-stable maximal tori and $H^1(F, W_{\mathbf{G}}(\mathbf{T}_0))$.

Example 4.18. Let $\mathfrak{X} = \{(\mathbf{T}, \mathbf{B}) \in \mathfrak{T} \times \mathfrak{B} \mid \mathbf{T} \leqslant \mathbf{B}\}$ then $\mathbf{G}$ acts transitively on $\mathfrak{X} \neq \varnothing$ by $g \cdot (\mathbf{T}, \mathbf{B}) = (g\mathbf{T}g^{-1}, g\mathbf{B}g^{-1})$. Taking $F' : \mathfrak{X} \to \mathfrak{X}$ to be the map $(\mathbf{T}, \mathbf{B}) \mapsto (F(\mathbf{T}), F(\mathbf{B}))$ it is easy to see that F' satisfies (a) above. Now, for any $(\mathbf{T}, \mathbf{B}) \in \mathfrak{X}$ we have

$$\mathrm{Stab}_{\mathbf{G}}(\mathbf{T}, \mathbf{B}) = N_{\mathbf{G}}(\mathbf{B}) \cap N_{\mathbf{G}}(\mathbf{T}) = \mathbf{B} \cap N_{\mathbf{G}}(\mathbf{T}) = N_{\mathbf{B}}(\mathbf{T}) = \mathbf{T}.$$

Note this last equality can be deduced from Lemma 3.47. In particular we have (b) holds and furthermore $\mathrm{Stab}_{\mathbf{G}}(\mathbf{T}, \mathbf{B})$ is connected.

By the previous example and Lemma 4.12 we can, and will, assume that our fixed maximal torus and Borel subgroup $\mathbf{T}_0 \leqslant \mathbf{B}_0$ are both F-stable. Note that this pair is then uniquely determined up to G-conjugacy.

Remark 4.19. Let us investigate a little bit further the F-stable maximal tori of $\mathbf{G}$. Given an element $w \in W_{\mathbf{G}}(\mathbf{T}_0)$ we fix a corresponding F-stable maximal torus $\mathbf{T}_w$ such that $\mathbf{T}_w = g\mathbf{T}_0 g^{-1}$ for some $g \in \mathbf{G}$ satisfying $\mathscr{L}(g) = \dot{w}$. We say $\mathbf{T}_w$ is an F-stable maximal torus *obtained from* $\mathbf{T}_0$ *by twisting with* w. The corresponding fixed point group under F is given by

$$\begin{aligned} T_w &= \{{}^g t \mid t \in \mathbf{T}_0 \text{ and } F({}^g t) = {}^g t\}, \\ &= \{{}^g t \mid t \in \mathbf{T}_0 \text{ and } F(t) = t^{\dot{w}}\} \end{aligned}$$

because $F(g) = g\dot{w}$. We define $\widehat{T}_w = {}^{g^{-1}}T_w$, which is the subgroup of $\mathbf{T}_0$ consisting of all elements satisfying $F(t) = t^{\dot{w}}$. Let us denote by $F_w : \mathbf{T}_0 \to \mathbf{T}_0$ the homomorphism given by $F_w(t) = {}^{\dot{w}}F(t)$ then if $F(\dot{w}) = \dot{w}$ this is a Frobenius endomorphism of $\mathbf{T}_0$ by Exercise 4.6. The conjugation map $\mathrm{Inn}_g : \mathbf{G} \to \mathbf{G}$ defines an isomorphism $\mathbf{T}_0 \to \mathbf{T}_w$ such that $F \circ \mathrm{Inn}_g = \mathrm{Inn}_g \circ F_w$, in particular this restricts to an isomorphism of finite groups $\widehat{T}_w \to T_w$. In general, we work with $\widehat{T}_w$ instead of T_w as it is difficult to construct explicitly an element $g \in \mathbf{G}$ satisfying $\mathscr{L}(g) = \dot{w}$.

Example 4.20. Assume $\mathbf{G} = \mathrm{GL}_3(\mathbb{K})$ and let $\mathbf{T}_0$ be the maximal torus consisting of diagonal matrices. We take $F = F_{q,\phi}$ to be the standard Frobenius endomorphism where $\phi : \mathrm{GL}_3(\mathbb{K}) \hookrightarrow \mathrm{Mat}_3(\mathbb{K})$ is the natural inclusion. Let

$$\dot{w} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in N_{\mathbf{G}}(\mathbf{T}_0)$$

then we have

$$F_w(\mathrm{diag}(t_1, t_2, t_3)) = \mathrm{diag}(t_2^q, t_1^q, t_3^q).$$

In particular this shows us that

$$\hat{T}_w = \mathbf{T}_0^{F_w} = \{\mathrm{diag}(t_1, t_1^q, t_3) \in \mathbf{T}_0 \mid t_1^{q^2-1} = t_3^{q-1} = 1\},$$

which is isomorphic to the direct product of cyclic groups $C_{q^2-1} \times C_{q-1}$.

Exercise 4.21. Construct a matrix $g \in \mathrm{GL}_3(\mathbb{K})$ such that $\mathscr{L}(g) = \dot{w}$, hence write down explicitly the subgroup T_w of $\mathrm{GL}_3(q)$.

∘ *Conjugacy Classes*

Assume we have obtained a full list of the conjugacy classes in $\mathbf{G}$. If $\mathcal{C}$ is an F-stable conjugacy class of $\mathbf{G}$ then $\mathbf{G}$ clearly acts transitively on $\mathcal{C}$. We define F' to be the restriction of F to $\mathcal{C}$ then this action and F' satisfy conditions (a) and (b) above. The orbits $\mathcal{C}^F/G$ are the *rational conjugacy classes* contained in $\mathcal{C}^F$. Clearly the stabiliser of a point $x_0 \in \mathcal{C}^F$ is just the centraliser $C_{\mathbf{G}}(x_0)$ so by Theorem 4.14 the rational classes $\mathcal{C}^F/G$ are in bijection with the F-conjugacy classes $H^1(F, A_{\mathbf{G}}(x_0))$ where $A_{\mathbf{G}}(x_0) = C_{\mathbf{G}}(x_0)/C_{\mathbf{G}}(x_0)^\circ$. In particular, if $C_{\mathbf{G}}(x_0)$ is connected then $\mathcal{C}^F$ is a single G-conjugacy class.

Every element $x \in G$ lies in an F-stable conjugacy class of $\mathbf{G}$. Furthermore, every F-stable conjugacy class of $\mathbf{G}$ contains an element of G. Hence to determine the conjugacy classes of G it is enough to determine the F-stable conjugacy classes $\mathcal{C}$ of $\mathbf{G}$ then use Theorem 4.14 to parameterise the rational classes in $\mathcal{C}^F/G$.

∘ *The Weyl group and Bruhat decomposition of G*

As $\mathbf{T}_0$ is F-stable we have F induces an automorphism of the Weyl group $W = W_{\mathbf{G}}(\mathbf{T}_0) = N_{\mathbf{G}}(\mathbf{T}_0)/\mathbf{T}_0$, which we again denote by F. Assume $\mathbb{S} = \{s_\alpha \mid \alpha \in \Delta(\mathbf{T}_0)\}$ is the set of simple reflections for W determined by $\mathbf{B}_0$ then, as $\mathbf{B}_0$ is F-stable, we have $F(\mathbb{S}) = \mathbb{S}$. Let us denote by $\mathbb{S}/F$ the orbits of F acting on $\mathbb{S}$. Assume $J \in \mathbb{S}/F$ is an F-orbit and let $W_J = \langle s \in J \rangle \leqslant W$ be the F-stable parabolic subgroup generated by J. Denote by $s_J \in W_J$ the longest element of W_J (c.f. Lemma 2.17). It is easy to check that $s_J \in W_J^F \leqslant W^F$.

Proposition 4.22. *Let $\mathbb{T} = \{s_J \mid J \in \mathbb{S}/F\}$ then $(W^F, \mathbb{T})$ is a Coxeter system (see Definition 2.14). In particular, W^F is a Coxeter group.*

Exercise 4.23. Show that if F is the identity on W then $\mathbb{T} = \mathbb{S}$. In particular, the Coxeter system $(W^F, \mathbb{T})$ is simply $(W, \mathbb{S})$.

Remark 4.24. It is usually the case that W^F is a Weyl group but this is not always the case. An example is given by the case where $\mathbf{G}$ is a simple group of type F_4, $p = 2$ and F is a generalised Frobenius endomorphism inducing the exceptional graph automorphism of W. In this situation W^F is isomorphic to a dihedral group of order 16 which is not a Weyl group. This is the only example when $\mathbf{G}$ is simple.

In spite of this remark we will call W^F the *Weyl group* of the finite reductive group G. This definition is somewhat justified by the following corollary of the Lang–Steinberg theorem.

Lemma 4.25. *Assume $\mathbf{H}$ is an affine algebraic group and $\mathbf{N} \leqslant \mathbf{H}$ is a closed* connected *normal subgroup then $(\mathbf{H}/\mathbf{N})^F \cong \mathbf{H}^F/\mathbf{N}^F$.*

Remark 4.26. It is important to note that if $\mathbf{N}$ is not connected then the above statement is no longer true!

In particular, we have $W^F = W_{\mathbf{G}}(\mathbf{T}_0)^F \cong N_{\mathbf{G}}(\mathbf{T}_0)^F/\mathbf{T}_0^F$. This implies that every element $w \in W_{\mathbf{G}}(\mathbf{T}_0)^F$ has a representative $\dot{w} \in N_{\mathbf{G}}(\mathbf{T}_0)^F$. For the algebraic group we noted that $W_{\mathbf{G}}(\mathbf{T})$ is isomorphic to $W_{\mathbf{G}}(\mathbf{T}_0)$ for any maximal torus $\mathbf{T}$ of $\mathbf{G}$. However, this is not the case for the finite group G as is shown by the following exercise.

Exercise 4.27. Let $\mathbf{T}_w$ be an F-stable maximal torus obtained from $\mathbf{T}_0$ by twisting with $w \in W_{\mathbf{G}}(\mathbf{T}_0)$. Prove that

$$W_{\mathbf{G}}(\mathbf{T}_w)^F \cong C_{W,F}(w) := \{x \in W_{\mathbf{G}}(\mathbf{T}_0) \mid x^{-1}wF(x) = w\}.$$

We call $C_{W,F}(w)$ the F-centraliser of $w \in W$. (Hint: use the identification of $\mathbf{T}_w$ equipped with F with $\mathbf{T}_0$ equipped with F_w to show that $W_{\mathbf{G}}(\mathbf{T}_w)^F \cong W_{\mathbf{G}}(\mathbf{T}_0)^{F_w}$.)

Now that we have the notion of the Weyl group in place we can obtain a direct analogue of Theorem 3.33 for the finite reductive group G.

Exercise 4.28. Assume $\mathbf{G}$ is a connected reductive algebraic group, F is a generalised Frobenius endomorphism of $\mathbf{G}$ and $\mathbf{T} \leqslant \mathbf{B}$ are respectively an F-stable maximal torus and F-stable Borel subgroup of $\mathbf{G}$. If $W = W_{\mathbf{G}}(\mathbf{T})$ is the corresponding Weyl group then we have

$$G = \bigsqcup_{w \in W^F} B\dot{w}B$$

where this union is disjoint and $\dot{w} \in N_{\mathbf{G}}(\mathbf{T})^F$ is a fixed representative of $w \in W_{\mathbf{G}}(\mathbf{T})^F$. (Hint: use Theorem 3.33).

◦ *Duality*

Recall the definition of a dual group for $\mathbf{G}$ given in Definition 3.61 and the notation introduced for $\mathbf{G}^\star$. We wish to now define what it means for a finite reductive group to be a dual group of G.

Definition 4.29. Assume $\mathbf{G}^\star$ is a dual group of $\mathbf{G}$ and let $\varphi : X(\mathbf{T}_0) \to \check{X}(\mathbf{T}_0^\star)$ be the isomorphism defining the duality. Assume $F^\star : \mathbf{G}^\star \to \mathbf{G}^\star$ is a generalised Frobenius endomorphism of $\mathbf{G}^\star$ then the fixed point group $G^\star$ is a *dual group* of G if $\mathbf{T}_0^\star$ and $\mathbf{B}_0^\star$ are $F^\star$-stable and $\varphi \circ F = F^\star \circ \varphi$.

Remark 4.30. In this situation, we sometimes say the two quadruples $(\mathbf{T}_0, \mathbf{B}_0, \mathbf{G}, F)$ and $(\mathbf{T}_0^\star, \mathbf{B}_0^\star, \mathbf{G}^\star, F^\star)$ are in duality.

Exercise 4.31. Show that $\mathrm{GL}_n(q)$ is a dual group of $\mathrm{GL}_n(q)$.

Lemma 4.32. *Every finite reductive group G has a dual group $G^\star$.*

Recall that in Lemma 3.63 we defined an anti-isomorphism $W_{\mathbf{G}}(\mathbf{T}_0) \to W_{\mathbf{G}^\star}(\mathbf{T}_0^\star)$ between the Weyl groups of $\mathbf{G}$ and $\mathbf{G}^\star$. We now wish to investigate the applications of this anti-isomorphism to the parameterisation of F-stable maximal tori described in Example 4.17. Let us denote by $\mathfrak{T}^\star$ the set of all $F^\star$-stable maximal tori of $\mathbf{G}^\star$. We fix a representative $\dot{w} \in N_{\mathbf{G}^\star}(\mathbf{T}_0^\star)$ for each element $w \in \mathbf{W}^\star$. As above we can fix an $F^\star$-stable maximal torus $\mathbf{T}_w^\star$ such that $\mathbf{T}_w^\star = g\mathbf{T}_0^\star g^{-1}$ for some $g \in \mathbf{G}^\star$ satisfying $\mathscr{L}(g) = \dot{w}$, (where here $\mathscr{L}$ is applied in $\mathbf{G}^\star$).

Exercise 4.33. Show that the anti-isomorphism $W_{\mathbf{G}}(\mathbf{T}_0) \to W_{\mathbf{G}^\star}(\mathbf{T}_0^\star)$ induces a bijection $H^1(F, W_{\mathbf{G}}(\mathbf{T}_0)) \to H^1(F^\star, W_{\mathbf{G}^\star}(\mathbf{T}_0^\star))$. In particular, the map $\mathbf{T}_w \mapsto \mathbf{T}_{w^\star}^\star$ defines a bijection $\mathfrak{T}/G \to \mathfrak{T}^\star/G^\star$.

Definition 4.34. Given any maximal torus $\mathbf{T} \in \mathfrak{T}$ we denote by $\mathbf{T}^\star \in \mathfrak{T}^\star$ a maximal torus such that the corresponding classes in $\mathfrak{T}/G$ and $\mathfrak{T}^\star/G^\star$ are in bijective correspondence. We call $\mathbf{T}$ and $\mathbf{T}^\star$ *dual maximal tori*.

5. Ordinary Representation Theory of Finite Reductive Groups

In this section we will be interested in the character theory of our finite reductive group G. Let ℓ be a prime different from p and let us fix an algebraic closure $\overline{\mathbb{Q}}_\ell$ of the field of ℓ-adic numbers $\mathbb{Q}_\ell$ (this is an algebraically closed field of characteristic 0). If X is any finite set then we denote by $\overline{\mathbb{Q}}_\ell[X]$, or simply $\overline{\mathbb{Q}}_\ell X$, the free $\overline{\mathbb{Q}}_\ell$-vector space whose elements consist of formal sums

$$\sum_{x\in X} \alpha_x x$$

with $\alpha_x \in \overline{\mathbb{Q}}_\ell$. If X is a group then this carries the structure of an algebra over $\overline{\mathbb{Q}}_\ell$.

For any finite group H we will take the statement "M is an H-module" to mean that M is a left $\overline{\mathbb{Q}}_\ell H$-module. Similarly, if K is also a finite group then we will take the statement "M is an (H,K)-bimodule" to mean that M is both a left $\overline{\mathbb{Q}}_\ell H$-module and right $\overline{\mathbb{Q}}_\ell K$-module with compatible actions. We will denote by H–**mod** the category of finitely generated, equivalently finite dimensional, H-modules. We will also denote by $\mathrm{Cent}(H)$ the $\overline{\mathbb{Q}}_\ell$-vector space of functions $f : H \to \overline{\mathbb{Q}}_\ell$ which are constant on the conjugacy classes of H.

Let us fix an involutive automorphism $^{-} : \overline{\mathbb{Q}}_\ell \to \overline{\mathbb{Q}}_\ell$ such that $\overline{\omega} = \omega^{-1}$ for all roots of unity $\omega \in \overline{\mathbb{Q}}_\ell^\times$. The space $\mathrm{Cent}(H)$ has a basis given by the set of irreducible characters $\mathrm{Irr}(H)$ which is orthonormal with respect to the usual inner product $\langle -,-\rangle_H : \mathrm{Cent}(H) \times \mathrm{Cent}(H) \to \overline{\mathbb{Q}}_\ell$ defined by

$$\langle f, f'\rangle_H = \frac{1}{|H|}\sum_{h\in H} f(h)\overline{f'(h)}.$$

If M and M' are H-modules then we define

$$\langle M, M'\rangle_H = \dim \mathrm{Hom}_{\overline{\mathbb{Q}}_\ell H}(M, M').$$

If f and f' are characters afforded respectively by modules M and M' then we have $\langle M, M'\rangle_H = \langle f, f'\rangle_H$.

We will also use $\mathrm{Irr}(H)$ to denote the isomorphism classes of simple H-modules, which are naturally in bijective correspondence with the irreducible characters. Note that the Grothendieck group $K_0(H$–**mod**$)$ may naturally be identified with the $\mathbb{Z}$-module of all virtual characters $\mathbb{Z}\,\mathrm{Irr}(H)$ (i.e. the set of all $\mathbb{Z}$-linear combinations of $\mathrm{Irr}(H)$) and the extension by scalars $\overline{\mathbb{Q}}_\ell \otimes_{\mathbb{Z}} K_0(H$–**mod**$)$ may be identified with $\mathrm{Cent}(H)$.

For a finite group H we will often need to consider the *opposite group* H^{opp}. As a set this is simply H endowed with a new multiplication $*$ given by $x * y = yx$ for all $x, y \in H^{\mathrm{opp}}$. Using the isomorphism $H \to H^{\mathrm{opp}}$ defined by $x \mapsto x^{-1}$ we will respectively identify $\mathrm{Irr}(H) \subseteq \mathbb{Z}\,\mathrm{Irr}(H) \subseteq \mathrm{Cent}(H)$ with $\mathrm{Irr}(H^{\mathrm{opp}}) \subseteq \mathbb{Z}\,\mathrm{Irr}(H^{\mathrm{opp}}) \subseteq \mathrm{Cent}(H^{\mathrm{opp}})$ without specific mention.

Remark 5.1. Typically when considering ordinary irreducible characters of a finite group, one would work over the complex numbers $\mathbb{C}$. However, we will see that to employ powerful techniques from algebraic geometry we will need to work over $\overline{\mathbb{Q}}_\ell$. One quick (but slightly undesirable) way to see that the objects are equivalent is to note that $\overline{\mathbb{Q}}_\ell$ and $\mathbb{C}$ are isomorphic as fields (assuming the axiom of choice). Under such an isomorphism one may also identify the involutive automorphism with complex conjugation.

◦ *Induction, Restriction and Inflation*

Recall that if H is any finite group and $K \leqslant H$ is a subgroup then we have two functors; the restriction functor $\mathrm{Res}_K^H : H\text{–}\mathbf{mod} \to K\text{–}\mathbf{mod}$ and the induction functor $\mathrm{Ind}_K^H : K\text{–}\mathbf{mod} \to H\text{–}\mathbf{mod}$. For $M \in K\text{–}\mathbf{mod}$ and $N \in H\text{–}\mathbf{mod}$ these are given by

$$\mathrm{Res}_K^H(M) = M \qquad \text{and} \qquad \mathrm{Ind}_K^H(N) = \overline{\mathbb{Q}}_\ell H \otimes_{\overline{\mathbb{Q}}_\ell K} N,$$

where in the restriction K simply acts through the H-action. These respectively induce $\overline{\mathbb{Q}}_\ell$-linear maps $\mathrm{Res}_K^H : \mathrm{Cent}(H) \to \mathrm{Cent}(K)$ and $\mathrm{Ind}_K^H : \mathrm{Cent}(K) \to \mathrm{Cent}(H)$. For any $\psi \in \mathrm{Cent}(K)$ we have

$$(\mathrm{Ind}_K^H)(\psi)(h) = \frac{1}{|K|}\sum_{x\in H} \dot{\psi}(xhx^{-1}),$$

for all $h \in H$, where $\dot{\psi}(y) = \psi(y)$ if $y \in K$ and 0 otherwise.

We will denote by H/K the set of left cosets $\{hK \mid h \in H\}$ of K in H and similarly by $K\backslash H$ the set of right cosets $\{Kh \mid h \in H\}$ of K in H. Assume now that K is normal of H and that there exists a subgroup $L \leqslant H$ such that $H = KL = LK$ and $K \cap L = \{1\}$ then we can define the inflation functor $\mathrm{Inf}_L^H : L\text{–}\mathbf{mod} \to H\text{–}\mathbf{mod}$ in the following way. The group algebra $\overline{\mathbb{Q}}_\ell[H/K]$ of the quotient group H/K is an (H, L^{opp})-bimodule where the actions are given by left and right multiplication. We then define for any L-module M the inflated module

$$\mathrm{Inf}_L^H(M) = \overline{\mathbb{Q}}_\ell[H/K] \otimes_{\overline{\mathbb{Q}}_\ell L} M.$$

Note that K acts trivially on this module. The inflation functor induces a $\overline{\mathbb{Q}}_\ell$-linear map $\mathrm{Inf}_L^H : \mathrm{Cent}(L) \to \mathrm{Cent}(H)$ given by $\mathrm{Inf}_L^H(\psi) = \psi \circ \pi$ where $\pi : H \to L$ is the natural projection map.

◦ *Harish-Chandra Induction and Restriction*

Let us assume that $\mathbf{P}$ is an F-stable parabolic subgroup of $\mathbf{G}$ with F-stable Levi complement $\mathbf{L}$. Recall that $\mathbf{P}$ has a semidirect product decomposition $\mathbf{L} \ltimes \mathbf{U}$, where $\mathbf{U} = R_u(\mathbf{P})$, then we have $\mathbf{U}$ is F-stable as both $\mathbf{P}$ and $\mathbf{L}$ are F-stable. In particular, we have a semidirect product decomposition $P = L \ltimes U$.

Definition 5.2. Assume $\mathbf{L}$ is an F-stable Levi subgroup of an F-stable parabolic subgroup

$\mathbf{P} = \mathbf{L} \ltimes \mathbf{U}$ then we define $R^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}} : L\text{–}\mathbf{mod} \to G\text{–}\mathbf{mod}$ to be the functor given by

$$R^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}}(M) = \overline{\mathbb{Q}}_\ell[G/U] \otimes_{\overline{\mathbb{Q}}_\ell L} M.$$

We call $R^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}}$ a *Harish-Chandra induction* functor. This admits an adjoint functor ${}^*R^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}} : G\text{–}\mathbf{mod} \to L\text{–}\mathbf{mod}$ known as a *Harish-Chandra restriction* functor given by

$${}^*R^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}}(M) = \overline{\mathbb{Q}}_\ell[U\backslash G] \otimes_{\overline{\mathbb{Q}}_\ell G} M.$$

Exercise 5.3. Prove that $R^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}}$ and ${}^*R^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}}$ are adjoint functors.

Exercise 5.4. Prove that $\overline{\mathbb{Q}}_\ell[G/U]$ and $\overline{\mathbb{Q}}_\ell G \otimes_{\overline{\mathbb{Q}}_\ell P} \overline{\mathbb{Q}}_\ell[P/U]$ are isomorphic as (G, L^{opp})-bimodules. Deduce that $R^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}}$ is simply the functor $\mathrm{Ind}_P^G \circ \mathrm{Inf}_L^P$.

We will see later that the following result can be deduced from the Mackey formula for Harish-Chandra induction and restriction. However, we recall this now for convenience.

Lemma 5.5. *Assume $\mathbf{P}$ and $\mathbf{P}'$ are F-stable parabolic subgroups of $\mathbf{G}$ having $\mathbf{L}$ as an F-stable Levi complement then for any L-module M and any G-module N we have*

$$R^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}}(M) \cong R^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}'}(M) \qquad\qquad {}^*R^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}}(N) \cong {}^*R^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}'}(N).$$

Consequently, we will simply write $R^{\mathbf{G}}_{\mathbf{L}}$ for the functor $R^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}}$.

To obtain functors L–**mod** $\to$ G–**mod** and G–**mod** $\to$ L–**mod** we could have simply considered the usual induction and restriction functors. However, these turn out to not have such desirable properties. In particular, given a simple module $M \in \mathrm{Irr}(L)$ the decomposition of $\mathrm{Ind}^{\mathbf{G}}_{\mathbf{L}}(M)$ into indecomposable submodules is quite intractable. To show that we are in a better situation for $R^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}}(M)$ we need to recall the *fitting correspondence*.

Let H be a finite group and M an H-module. Let us denote by $E = \mathrm{End}_{\overline{\mathbb{Q}}_\ell H}(M)^{\mathrm{opp}}$ the opposite algebra of the endomorphism algebra of M. We then have a functor $\mathfrak{F}_M : H\text{–}\mathbf{mod} \to E\text{–}\mathbf{mod}$ given by $\mathfrak{F}_M(V) = \mathrm{Hom}_{\overline{\mathbb{Q}}_\ell H}(M, V)$. Here E–**mod** is the category of all finitely generated left E-modules and E acts on $\mathfrak{F}_M(X)$ by precomposition. Let $\mathrm{Irr}(H \mid M) \subseteq \mathrm{Irr}(H)$ be all isomorphism classes of simple modules which are isomorphic to a submodule of M and let $\mathrm{Irr}(E)$ be the set of all isomorphism classes of simple E-modules.

Proposition 5.6 (Fitting correspondence). *The functor $\mathfrak{F}_M$ induces a bijection*

$$\mathfrak{F}_M : \mathrm{Irr}(H \mid M) \to \mathrm{Irr}(E).$$

Assume now that $M = M_1 \oplus \cdots \oplus M_k$ is a decomposition into indecomposable H-submodules and set $E_i = \mathfrak{F}_M(M_i)$ for each $1 \leqslant i \leqslant k$ then the following hold:

(i) $E = E_1 \oplus \cdots \oplus E_k$

(ii) *for any* $1 \leqslant i, j \leqslant k$ *we have* $M_i \cong M_j$ *as* H*-modules if and only if* $E_i \cong E_j$ *as* E*-modules.*

The fitting correspondence tells us that giving a decomposition of $R_{\mathbf{L}}^{\mathbf{G}}(M)$ for a simple module $M \in \mathrm{Irr}(L)$ is equivalent to giving a decomposition of its endomorphism algebra. In particular, this leads us to the question: what is the endomorphism algebra of $R_{\mathbf{L}}^{\mathbf{G}}(M)$? The answer comes in the form of Hecke algebras but before we go into this we first consider an inductive parameterisation of $\mathrm{Irr}(G)$ given by Harish-Chandra induction.

Definition 5.7. A simple module $M \in \mathrm{Irr}(G)$ is said to be *cuspidal* if ${}^*R_{\mathbf{L}}^{\mathbf{G}}(M) = 0$ for every F-stable Levi complement $\mathbf{L} \neq \mathbf{G}$ of an F-stable parabolic subgroup.

Remark 5.8. Note that if G is a torus then every simple module is cuspidal.

Proposition 5.9. *Assume now that* $M \in \mathrm{Irr}(G)$ *is a simple* G*-module then the following hold:*

(i) *there exists a pair* $(\mathbf{L}, N)$ *where:* $\mathbf{L}$ *is an* F*-stable Levi complement of an* F*-stable parabolic subgroup of* $\mathbf{G}$ *and* $N \in \mathrm{Irr}(L)$ *is a cuspidal simple module such that* $\langle R_{\mathbf{L}}^{\mathbf{G}}(N), M\rangle_G \neq 0$.

(ii) *assume* $(\mathbf{L}, N)$ *and* $(\mathbf{L}', N')$ *are minimal such that* $\langle R_{\mathbf{L}}^{\mathbf{G}}(N), M\rangle \neq 0$ *and* $\langle R_{\mathbf{L}}^{\mathbf{G}}(N), M\rangle \neq 0$ *then there exists* $g \in G$ *such that* $({}^g\mathbf{L}, {}^gN) = (\mathbf{L}', N')$.

This shows that we have an inductive method for parameterising $\mathrm{Irr}(G)$ which depends on the following two steps:

- classify the cuspidal simple modules of any finite reductive group,
- determine the structure of the endomorphism algebra $\mathrm{End}_{\overline{\mathbb{Q}}_\ell G}(R_{\mathbf{L}}^{\mathbf{G}}(N))^{\mathrm{opp}}$ for any cuspidal simple module $N \in \mathrm{Irr}(L)$.

The first part turns out to be quite difficult but was achieved by Lusztig using Deligne–Lusztig induction and restriction (which we will meet later). What we will discuss now is the second problem in the special case where $\mathbf{L}$ is a maximal torus and N is the trivial module.

∘ *Hecke algebras*

Let $\mathbf{B} = \mathbf{T} \ltimes \mathbf{U}$ be an F-stable Borel subgroup such that $\mathbf{T}$ is an F-stable maximal torus. We will denote by $W = W_{\mathbf{G}}(\mathbf{T})$ the Weyl group of $\mathbf{G}$ with respect to $\mathbf{T}$ and $\mathbb{S} \subseteq W$ the set of simple reflections determined by $\mathbf{B}$. We will denote by $(W^F, \mathbb{T})$ the Coxeter system described in Proposition 4.22. Consider the submodule $M = \overline{\mathbb{Q}}_\ell Be \subset \overline{\mathbb{Q}}_\ell B$ generated by the idempotent

$$e := \frac{1}{|B|}\sum_{b \in B} b,$$

then M is isomorphic to the trivial module of B. We define the *Hecke algebra* $\mathcal{H}(G, B)$ to be the $\overline{\mathbb{Q}}_\ell$-algebra $e\overline{\mathbb{Q}}_\ell Ge$. Note that $e\overline{\mathbb{Q}}_\ell Ge \subset \overline{\mathbb{Q}}_\ell G$ is not a subalgebra as it does not contain the identity.

Exercise 5.10. Check that $\mathrm{Ind}_B^G(M)$ is isomorphic to $(\mathrm{Ind}_B^G \circ \mathrm{Inf}_T^B)(M') \cong R_{\mathbf{T}}^{\mathbf{G}}(M')$ where $M' \in \mathrm{Irr}(T)$ is also the trivial module.

Recall that $e\overline{\mathbb{Q}}_\ell Ge$ is isomorphic to $\mathrm{End}_{\overline{\mathbb{Q}}_\ell G}(M)^{\mathrm{opp}}$, hence the Hecke algebra $\mathcal{H}(G, B)$ is isomorphic to our desired endomorphism algebra. For any $w \in W^F$ we will write $\dot{w} \in N_{\mathbf{G}}(\mathbf{T})^F$ for a representative of w (which exists by Lemma 4.25) and define

$$\bar{T}_w := \frac{1}{|B|} \sum_{x \in B\dot{w}B} x.$$

We call $\bar{T}_w$ a standard basis element of $\mathcal{H}(G, B)$. The Hecke algebra is generated by the set $\{\bar{T}_s \mid s \in \mathbb{T}\}$ and the basis elements satisfy the relations

$$\begin{aligned} \bar{T}_s\bar{T}_w &= \bar{T}_{sw} && \text{if } \ell(sw) > \ell(w), \\ \bar{T}_s\bar{T}_w &= q_s\bar{T}_{sw} + (q_s - 1)\bar{T}_w && \text{if } \ell(sw) < \ell(w), \end{aligned}$$

for any $s \in \mathbb{T}$ and $w \in W^F$, where $q_s := [B : \dot{s}B\dot{s}^{-1} \cap B] = q^{c_s}$ is the *index parameter*. Note that, here, the length function ℓ is that of the Coxeter group W^F. If s is W^F-conjugate to s' then we have $q_s = q_{s'}$ or equivalently $c_s = c_{s'}$.

Exercise 5.11. Assume $\mathbf{G} = \mathrm{GL}_n(\mathbb{K})$, $\mathbf{T}$ is the subgroup of diagonal matrices, $\mathbf{B}$ is the subgroup of upper triangular matrices F is the standard Frobenius endomorphism $F_{q,\phi}$ where $\phi : \mathrm{GL}_n(\mathbb{K}) \to \mathrm{Mat}_n(\mathbb{K})$ is the natural inclusion map. Show that $q_s = q$ for all simple reflections $s \in \mathbb{S}$.

Remark 5.12. In general if F is a Frobenius endomorphism defining an $\mathbb{F}_q$-rational structure on $\mathbf{G}$ and F induces the identity on W then $q_s = q$ for all $s \in \mathbb{T} = \mathbb{S}$.

This algebra quite clearly depends on the index parameters q_s. What we would like to do is study the Hecke algebra independently of these parameters.

Definition 5.13. Let $(\mathcal{W}, \mathbb{J})$ be a Coxeter system and denote by R the commutative polynomial ring $\mathbb{Q}[u_j \mid j \in \mathbb{J}]$. We define the *generic Hecke algebra* $\mathcal{H}(\mathcal{W}, \mathbb{J})$ to be an R-algebra with basis $\{T_w \mid w \in \mathcal{W}\}$ whose elements satisfy the relations

$$\begin{aligned} T_jT_w &= T_{jw} && \text{if } \ell(jw) > \ell(w), \\ T_jT_w &= u_jT_{jw} + (u_j - 1)T_w && \text{if } \ell(jw) < \ell(w). \end{aligned}$$

for all $j \in \mathbb{J}$, $w \in \mathcal{W}$. Here ℓ is the length function of $\mathcal{W}$.

Remark 5.14. It can be shown, as before, that this algebra is generated by the set $\{T_j \mid j \in \mathbb{J}\}$.

We will assume now that $R = \mathbb{Q}[u_s \mid s \in \mathbb{T}]$ and that $\mathcal{H} := \mathcal{H}(W^F, \mathbb{T})$. We now consider how the generic Hecke algebra $\mathcal{H}$ relates to the Hecke algebra we first introduced.

Definition 5.15. A *specilisation* of R is a ring homomorphism $f : R \to \mathbb{F}$, where $\mathbb{F}$ is a field. If $f : R \to \mathbb{F}$ is a specialisation of R then $\mathbb{F}$ becomes an $(\mathbb{F}, R)$-bimodule with R acting on the right by f. We can then consider the tensor product $\mathcal{H}_f = \mathbb{F} \otimes_R \mathcal{H}$, which we call a *specialisation of* $\mathcal{H}$.

Proposition 5.16. *Let $f_1 : R \to \overline{\mathbb{Q}}_\ell$, $f_q : R \to \overline{\mathbb{Q}}_\ell$ be specilisations such that $f_1(u_s) = 1$ and $f_q(u_s) = q_s$ for all $s \in \mathbb{T}$. Then we have $\mathcal{H}_1 := \mathcal{H}_{f_1} \cong \overline{\mathbb{Q}}_\ell W^F$ and $\mathcal{H}_q := \mathcal{H}_{f_q} \cong \mathcal{H}(G, B)$ are both semisimple algebras.*

The isomorphisms are given by the canonical maps, namely $\sigma_1(w) = 1 \otimes T_w$ and $\sigma_q(\bar{T}_w) = 1 \otimes T_w$ for all $w \in W^F$. We would now like to relate the two specialisations given in the above proposition. Let K denote the field of fractions of the polynomial ring R then we can define the K-algebra $\tilde{\mathcal{H}} = K \otimes_R \mathcal{H}$. Before stating the required result we need to first introduce a definition.

Definition 5.17. Let A be an $\mathbb{F}$-algebra, where $\mathbb{F}$ is a field and let $\overline{\mathbb{F}}$ denote the algebraic closure of $\mathbb{F}$. The *numerical invariants* of A are then the dimensions of the simple modules of the $\overline{\mathbb{F}}$-algebra $A^\star = \overline{\mathbb{F}} \otimes_{\mathbb{F}} A$.

Theorem 5.18 (Tits's Deformation Theorem). *Let $f : R \to \overline{\mathbb{Q}}_\ell$ be a specilisation of R then $\tilde{\mathcal{H}}$ and $\mathcal{H}_f$ have the same numerical invariants.*

Corollary 5.19. *We have $\overline{\mathbb{Q}}_\ell W^F$ and $\mathcal{H}(G, B)$ are isomorphic as $\overline{\mathbb{Q}}_\ell$-algebras.*

Exercise 5.20. Prove Corollary 5.19. (Hint: use the Artin–Wedderburn theorem.)

Note that Corollary 5.19 does not give an explicit isomorphism between these two algebras. However, an explicit isomorphism is known and was constructuted by Lusztig, (see [Lus81, Theorem 3.1]). Using Corollary 5.19 we may now obtain our main result.

Theorem 5.21. *There is a bijection $\mathrm{Irr}(W^F) \to \mathrm{Irr}(G \mid R_{\mathbf{T}}^{\mathbf{G}}(M))$ which we denote $\rho \mapsto M_\rho$ such that*

$$\langle R_{\mathbf{T}}^{\mathbf{G}}(M), M_\rho \rangle_G = \dim \rho.$$

Example 5.22. Assume $\mathbf{G} = \mathrm{GL}_n(\mathbb{K})$ and F is the standard Frobenius endomorphism $F_{q,\phi}$ where $\phi : \mathrm{GL}_n(\mathbb{K}) \hookrightarrow \mathrm{Mat}_n(\mathbb{K})$ is the natural inclusion. We may then take $\mathbf{T}$ and $\mathbf{B}$ to be respectively the subgroup of all diagonal matrices and the subgroup of all upper triangular matrices. The automorpism of $W = W_{\mathbf{G}}(\mathbf{T}_0)$ induced by F is simply the identity, hence $W^F = W \cong \mathfrak{S}_n$. In particular, the set $\mathrm{Irr}(G \mid R_{\mathbf{T}}^{\mathbf{G}}(M))$ is in bijection with $\mathrm{Irr}(\mathfrak{S}_n)$ which is, in turn, in bijection with all partitions of n. This gives us a very explicit parameterisation of these simple modules. Furthermore the multiplicities are given by the dimensions of the simple modules for $\mathfrak{S}_n$, which can be computed using the hook-length formula.

Remark 5.23. The multiplicity condition in Theorem 5.21 does not uniquely determine the bijection. However, assuming $\mathbf{G}$ is simple then the bijection can be made unique in almost all cases using the specialisation maps. We will not go into this in detail here but simply refer the reader to [CR87, Theorem 68.26].

∘ *Kazhdan–Lusztig Cells*

Let $(\mathcal{W}, \mathbb{J})$ be a Coxeter system. We will denote by $\mathcal{H} = \mathcal{H}(\mathcal{W}, \mathbb{J})$ the generic Hecke algebra defined as in Definition 5.13. However, we will further assume that $u_j = u_k$ for all $j, k \in \mathbb{J}$ and simply denote this common variable by u. Additionally we will assume that $\mathcal{H}$ is defined over the Laurent polynomial ring $A := \mathbb{Z}[u^{1/2}, u^{-1/2}]$, not $\mathbb{Q}[u]$. We do this so that we can invert the standard basis elements T_w in $\mathcal{H}$, i.e. for $j \in \mathbb{J}$ we have

$$T_j^2 = uT_1 + (u-1)T_j \Rightarrow T_j^{-1} = (u^{-1} - 1)T_1 + u^{-1}T_j.$$

The Laurent polynomial ring A has a natural involutive automorphism denoted $\overline{} : A \to A$ which satisfies $\overline{u^{1/2}} = u^{-1/2}$ and $\overline{u^{-1/2}} = u^{1/2}$. We can then extend this to a ring automorphism $\overline{} : \mathcal{H} \to \mathcal{H}$ by setting

$$\overline{\sum_{w \in \mathcal{W}} c_w T_w} = \sum_{w \in \mathcal{W}} \overline{c_w} T_{w^{-1}}^{-1},$$

where $c_w \in A$. Note that this is not a linear map. Denote by $\leqslant$ the Bruhat ordering on $\mathcal{W}$ (see Definition 3.34) then we have the following fundamental result of Kazhdan and Lusztig.

Theorem 5.24 (Kazhdan and Lusztig). *For each element $w \in \mathcal{W}$ there exists a unique element $C_w \in \mathcal{H}$ such that $\overline{C_w} = C_w$ and*

$$C_w = \sum_{y \leqslant w} (-1)^{\ell(w)+\ell(y)} u^{\frac{1}{2}\ell(w)} u^{-\ell(y)} \overline{P_{y,w}(u)} T_y.$$

where $P_{w,w}(u) = 1$ and $P_{y,w}(u) \in \mathbb{Z}[u]$ has degree $\leqslant \frac{1}{2}(\ell(w) - \ell(y) - 1)$ if $y < w$. The polynomials $P_{y,w}(u)$ are called Kazhdan-Lusztig polynomials *and the set $\{C_w \mid w \in \mathbf{W}\}$ forms a basis for $\mathcal{H}$ called the* Kazhdan-Lusztig basis.

The Kazhdan-Lusztig basis is the key for defining the notion of cells for $\mathcal{W}$. If $x, y \in \mathcal{W}$ then we define $h_{x,y,z} \in A$ to be the structure constants given by

$$C_x C_y = \sum_{z \in \mathbf{W}} h_{x,y,z} C_z.$$

We write $z \leftarrow_{\mathscr{L}} y$ if there exists $x \in \mathcal{W}$ such that $h_{x,y,z} \neq 0$, (i.e. C_z appears in the product $C_x C_y$), then we write $\leqslant_{\mathscr{L}}$ for the pre-order relation on $\mathcal{W}$ generated by $\leftarrow_{\mathscr{L}}$. In other words we say $z \leqslant_{\mathscr{L}} y$ if there exists a sequence $z = y_0, y_1, \ldots, y_m = y$ such that $y_{i-1} \leftarrow_{\mathscr{L}} y_i$. We then define an associated equivalence relation $\sim_{\mathscr{L}}$ on $\mathcal{W}$ by setting $z \sim_{\mathscr{L}} y$ if $z \leqslant_{\mathscr{L}} y$ and $y \leqslant_{\mathscr{L}} z$. We denote the set of associated equivalence classes, called the *left cells* of $\mathcal{W}$, by $\mathcal{W}/\mathscr{L}$.

Similarly we write $z \leftarrow_{\mathscr{R}} y$ if there exists $x \in \mathcal{W}$ such that $h_{y,x,z} \neq 0$, (i.e. C_z appears in the product $C_y C_x$), then we write $\leqslant_{\mathscr{R}}$ for the associated pre-order and $\sim_{\mathscr{R}}$ for the associated equivalence relation. We denote the set of associated equivalence classes, called the right cells of $\mathcal{W}$, by $\mathcal{W}/\mathscr{R}$. Using the antihomomorphism $C_w \mapsto C_{w^{-1}}$ we see that $z \leqslant_{\mathscr{L}} y \Leftrightarrow z^{-1} \leqslant_{\mathscr{R}} y^{-1}$. In particular if $\mathcal{L} \subseteq \mathcal{W}$ if a left cell of $\mathcal{W}$ then the map $\mathcal{L} \mapsto \mathcal{L}^{-1}$, where

$$\mathcal{L}^{-1} := \{x^{-1} \mid x \in \mathcal{L}\},$$

gives a bijection between the left and right cells of $\mathcal{W}$.

Finally we can define a pre-order $\leqslant_{\mathscr{L}\mathscr{R}}$ on $\mathcal{W}$ by specifying that $z \leqslant_{\mathscr{L}\mathscr{R}} y$ if there exists a sequence $z = y_0, y_1, \ldots, y_m = y$ such that, for each $i \in \{1, \ldots, m\}$, we have $y_{i-1} \leqslant_{\mathscr{L}} y_i$ or $y_{i-1} \leqslant_{\mathscr{R}} y_i$. The equivalence relation associated with $\leqslant_{\mathscr{L}\mathscr{R}}$ is denoted by $\sim_{\mathscr{L}\mathscr{R}}$ and the corresponding set of equivalence classes, called the *two-sided cells* of $\mathcal{W}$, are denoted by $\mathcal{W}/\mathscr{L}\mathscr{R}$.

To determine the cells of $\mathcal{W}$ it is clear that we will have to determine products of Kazhdan-Lusztig basis elements. The reason why cells are introduced with respect to the basis $\{C_w \mid w \in \mathcal{W}\}$ and not the standard basis $\{T_w \mid w \in \mathcal{W}\}$ is due to the following result regarding multiplication.

Theorem 5.25 (Kazhdan and Lusztig). *Let $j \in \mathbb{J}$ be a simple reflection then for any $w \in \mathcal{W}$ we have*

$$C_j C_w = \begin{cases} C_{jw} + \displaystyle\sum_{y \in \mathcal{W}: jy < y < w} \mu(y, w) C_y & \text{if } jw > w, \\ -(u^{\frac{1}{2}} + u^{-\frac{1}{2}}) C_w & \text{if } jw < w, \end{cases}$$

$$C_w C_j = \begin{cases} C_{wj} + \displaystyle\sum_{z \in \mathcal{W}: zj < z < w} \mu(z^{-1}, w^{-1}) C_z & \text{if } wj > w, \\ -(u^{\frac{1}{2}} + u^{-\frac{1}{2}}) C_w & \text{if } wj < w, \end{cases}$$

where $\mu(y,w)$ is the coefficient of $u^{\frac{1}{2}(\ell(w)-\ell(y)-1)}$ in $P_{y,w}(u)$.

In particular, this theorem tells us that we can compute the left, right and two-sided cells if we can compute the Kazhdan-Lusztig polynomials.

Example 5.26. We consider the case where $(\mathcal{W},\mathbb{J})$ is the dihedral group of order 8 with $\mathbb{J} = \{a,b\}$. Recall that the Bruhat order for $\mathcal{W}$ is described in Example 3.36. It is known that, when $\mathcal{W}$ is a dihedral group we have $P_{y,w}(u) = 1$ for all $y,w \in \mathcal{W}$. In particular, we have

$$\mu(y,w) = \mu(y^{-1},w^{-1}) = \begin{cases} 1 & \text{if } \ell(w) = \ell(y)+1, \\ 0 & \text{otherwise.} \end{cases}$$

It is then clear from the multiplication relations of the Kazhdan-Lusztig basis that

$$abab \leftarrow_{\mathscr{L}} bab \leftarrow_{\mathscr{L}} ab \leftarrow_{\mathscr{L}} b \leftarrow_{\mathscr{L}} 1.$$

To determine the left cells we must reverse the arrows in this sequence.

Let $w_0 \in \mathcal{W}$ denote the longest element then it is easy to see that we always have $w_0 \leqslant_{\mathscr{L}} x$ and $w_0 \leqslant_{\mathscr{R}} x$ for any element $x \in \mathcal{W}$ because w_0 is maximal in the Bruhat ordering. Conversely, assume $j \in \mathbb{J}$ then we have $jw_0 < w_0$ and $w_0 j < w_0$ so

$$C_j C_{w_0} = C_{w_0} C_j = -(u^{\frac{1}{2}} + u^{-\frac{1}{2}})C_{w_0}.$$

In particular $x \leftarrow_{\mathscr{L}} w_0$ or $x \leftarrow_{\mathscr{R}} w_0 \Rightarrow x = w_0$ so $\{w_0\}$ must be a left, right and two-sided cell. Similar arguments show that $\{1\}$ is a left, right and two-sided cell.

We now progress down the chain of elements considered above. Making sensible choices at each stage we have

$$\begin{aligned} C_a C_{bab} &= C_{abab} + \mu(ab,bab)C_{ab} + \mu(a,bab)C_a, \\ &= C_{abab} + C_{ab}, \\ C_b C_{ab} &= C_{bab} + \mu(b,ab)C_b, \\ &= C_{bab} + C_b. \end{aligned}$$

Using these calculations it is clear that we also have a reverse sequence

$$b \leftarrow_{\mathscr{L}} ab \leftarrow_{\mathscr{L}} bab,$$

which tells us that $\{b,ba,bab\}$ forms a left cell of $\mathcal{W}$. An almost identical calculation shows that there is one remaining left cell and it is given by $\{a,ab,aba\}$. Hence the complete set of left cells is

$$\mathcal{W}/\mathscr{L} = \{1\} \sqcup \{a,ab,aba\} \sqcup \{b,ba,bab\} \sqcup \{abab\}.$$

Using the bijection between left cells and right cells we easily determine the complete set of right cells to be

$$\mathcal{W}/\mathscr{R} = \{1\} \sqcup \{a, ba, aba\} \sqcup \{b, ab, bab\} \sqcup \{abab\}.$$

In this case it is then easy to verify that the complete set of two sided cells is

$$\mathcal{W}/\mathscr{L}\mathscr{R} = \{1\} \sqcup \{a, ba, aba, b, ab, bab\} \sqcup \{abab\}.$$

◦ *Deligne–Lusztig Induction and Restriction*

What we would like to do is extend the Harish-Chandra functors $R^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}}$ and ${}^*R^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}}$ to the case where $\mathbf{P}$ is any parabolic subgroup (not necessarily F-stable) with F-stable Levi complement $\mathbf{L}$. This was achieved by Deligne and Lusztig in their landmark paper [DL76]. Unfortunately one does not obtain functors on the module categories but instead $\overline{\mathbb{Q}}_\ell$-linear maps between character rings. By work of Rickard and Rouquier, one can achieve functors by passing to the bounded derived category of the module category. However, to maintain simplicity we will work exclusively with virtual characters.

We will now describe Deligne and Lusztig's construction which uses, in a crucial way, the machinery provided by ℓ-adic cohomology. Let $\mathbf{X}$ be an affine algebraic variety over $\mathbb{K}$. For each $i \in \mathbb{Z}$ we have a finite dimensional $\overline{\mathbb{Q}}_\ell$-vector space $H^i_c(\mathbf{X}) := H^i_c(\mathbf{X}, \overline{\mathbb{Q}}_\ell)$ called the *ith ℓ-adic cohomology group with compact support*. The following theorem encapsulates some of the properties of $H^i_c(\mathbf{X})$.

Theorem 5.27. *Assume $\mathbf{X}$ is an affine algebraic variety over $\mathbb{K}$ and $i \in \mathbb{Z}$ then the following hold.*

(i) $H^i_c(\mathbf{X}) = 0$ *if* $i \notin \{0, \dots, 2\dim \mathbf{X}\}$.

(ii) Any finite morphism $F : \mathbf{X} \to \mathbf{X}$ induces a linear endomorphism $F^* : H^i_c(\mathbf{X}) \to H^i_c(\mathbf{X})$ for any $i \in \mathbb{Z}$ and this correspondence is functorial. If F is a Frobenius endomorphism then F^* is an automorphism.

(iii) If $g \in \mathrm{Aut}(\mathbf{X})$ *is a finite automorphism of* $\mathbf{X}$ *then the* Lefschetz number

$$\mathcal{L}(g, \mathbf{X}) = \sum_{i \geqslant 0} (-1)^i \operatorname{tr}(g, H^i_c(\mathbf{X}))$$

of g acting on $\mathbf{X}$ is an integer independent of ℓ.

(iv) Assume $\mathbf{Y}$ is also an affine algebraic variety over $\mathbb{K}$ and $f : \mathbf{X} \to \mathbf{Y}$ is a bijective morphism. If $g \in \mathrm{Aut}(\mathbf{X})$, $g' \in \mathrm{Aut}(\mathbf{Y})$ *are finite morphisms such that* $f \circ g = g' \circ f$ *then*

$$\mathcal{L}(g, \mathbf{X}) = \mathcal{L}(g', \mathbf{Y}).$$

(v) If $F : \mathbf{X} \to \mathbf{X}$ is a Frobenius endomorphism then

$$|\mathbf{X}^F| = \sum_{i\in\mathbb{Z}} (-1)^i \operatorname{Tr}(F^*, H^i_c(\mathbf{X})).$$

Remark 5.28. Note that the functoriality of ℓ-adic cohomology mentioned in (ii) of Theorem 5.27 only holds for *finite morphisms*. The reader can be assured that we will only use this in the context of finite morphisms.

Let $\mathbf{U} = R_u(\mathbf{P})$ be the unipotent radical of $\mathbf{P}$ then, as $\mathbf{P}$ is not necessarily F-stable, we do not necessarily have that $\mathbf{U}$ is F-stable. We define

$$\mathbf{Y}^{\mathbf{G}}_{\mathbf{U}} = \mathscr{L}^{-1}(\mathbf{U}) = \{g \in \mathbf{G} \mid g^{-1}F(g) \in \mathbf{U}\},$$

which is an algebraic subset of $\mathbf{G}$. For each $g \in G$, $l \in L$ and for all $x \in \mathscr{L}^{-1}(\mathbf{U})$ we have

$$\begin{aligned} \mathscr{L}(gx) &= x^{-1}g^{-1}F(g)F(x) = x^{-1}F(x) \in \mathbf{U}, \\ \mathscr{L}(xl) &= l^{-1}x^{-1}F(x)F(l) = l^{-1}x^{-1}F(x)l \in l^{-1}\mathbf{U}l = \mathbf{U}. \end{aligned}$$

In particular, the direct product $G \times L^{\mathrm{opp}}$ acts on $\mathbf{Y}^{\mathbf{G}}_{\mathbf{U}}$ as a group of finite automorphisms via $(g,l)\cdot x = gxl$. By Theorem 5.27 this action induces an action of $G \times L^{\mathrm{opp}}$ on $H^i_c(\mathbf{Y}^{\mathbf{G}}_{\mathbf{U}})$ which makes $H^i_c(\mathbf{Y}^{\mathbf{G}}_{\mathbf{U}})$ a (G, L^{opp})-bimodule. Similarly, by exchanging the factors in the direct product we have $H^i_c(\mathbf{Y}^{\mathbf{G}}_{\mathbf{U}})$ is an (L^{opp}, G)-bimodule.

Definition 5.29. Assume $\mathbf{P} = \mathbf{L} \ltimes \mathbf{U}$ is a parabolic subgroup with F-stable Levi complement $\mathbf{L}$ then we define $R^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}} : \operatorname{Irr}(L) \to \mathbb{Z}\operatorname{Irr}(G)$ by setting

$$R^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}}(\theta)(g) = \sum_{i\geqslant 0} (-1)^i \operatorname{Tr}(g, H^i_c(\mathbf{Y}^{\mathbf{G}}_{\mathbf{U}}) \otimes_{\overline{\mathbb{Q}}_\ell L} \theta),$$

for all $g \in G$. We call $R^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}}$ a *Deligne–Lusztig induction map*. Similarly we have a map ${}^*R^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}} : \operatorname{Irr}(G) \to \mathbb{Z}\operatorname{Irr}(L)$ defined by setting

$${}^*R^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}}(\theta)(l) = \sum_{i\geqslant 0} (-1)^i \operatorname{Tr}(l, H^i_c(\mathbf{Y}^{\mathbf{G}}_{\mathbf{U}}) \otimes_{\overline{\mathbb{Q}}_\ell G} \chi),$$

for all $l \in L$. We call ${}^*R^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}}$ a *Deligne–Lusztig restrcition map*.

Remark 5.30. By linearity we can extend the Deligne–Lusztig induction and restriction maps $R^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}}$ and ${}^*R^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}}$ to maps $\operatorname{Cent}(L) \to \operatorname{Cent}(G)$ and $\operatorname{Cent}(G) \to \operatorname{Cent}(L)$ respectively.

Lemma 5.31. *The Deligne–Lusztig induction and restriction maps satisfy Frobenius reciprocity. In* *particular, for any* $\theta \in \mathrm{Cent}(L)$ *and* $\chi \in \mathrm{Cent}(G)$ *we have*

$$\langle R^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}}(\theta), \chi\rangle_G = \langle \theta, {}^*R^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}}(\chi)\rangle_L.$$

If $\mathbf{P}$ is F-stable then we have used $R^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}}$ and ${}^*R^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}}$ to stand for both Deligne–Lusztig induction and restriction and Harish-Chandra induction and restriction. However, the following shows that when $\mathbf{P}$ is F-stable these notions coincide.

Proposition 5.32. *Assume* $\mathbf{P}$ *is* F*-stable then for any* $i \in \mathbb{Z}$ *we have*

$$H^i_c(\mathbf{Y}^{\mathbf{G}}_{\mathbf{U}}) = \begin{cases} 0 & \text{if } i \neq 2\dim \mathbf{U} \\ \overline{\mathbb{Q}}_\ell[G/U] & \text{if } i = 2\dim \mathbf{U} \end{cases}$$

Exercise 5.33. Using the above result confirm that $R^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}}$ and ${}^*R^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}}$ coincide with the Harish-Chandra induction and restriction maps defined in Definition 5.2 when $\mathbf{P}$ is F-stable.

We noted in Lemma 5.5 that Harish-Chandra induction and restriction functors were independent of the parabolic subgroup used to define them. We will see now that this follows from the validity of the *Mackey formula*.

Assume $\mathbf{P}$ and $\mathbf{Q}$ are parabolic subgroups with F-stable Levi complements $\mathbf{L}$ and $\mathbf{M}$ respectively. We denote by $\mathcal{S}_{\mathbf{G}}(\mathbf{L},\mathbf{M})$ the set of all $g \in \mathbf{G}$ such that $\mathbf{L} \cap {}^g\mathbf{M}$ contains a maximal torus of $\mathbf{G}$. We say the *Mackey formula holds* for $\mathbf{L} \subseteq \mathbf{P}$ and $\mathbf{M} \subseteq \mathbf{Q}$ if the following equality holds

$$ {}^*R^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}} \circ R^{\mathbf{G}}_{\mathbf{M}\subseteq\mathbf{Q}} = \sum_{g\in[L\backslash\mathcal{S}_{\mathbf{G}}(\mathbf{L},\mathbf{M})^F/M]} R^{\mathbf{L}}_{\mathbf{L}\cap{}^g\mathbf{M}\subseteq\mathbf{L}\cap{}^g\mathbf{Q}} \circ {}^*R^{{}^g\mathbf{M}}_{\mathbf{L}\cap{}^g\mathbf{M}\subseteq\mathbf{P}\cap{}^g\mathbf{M}} \circ (\mathrm{ad}\, g)_M. \qquad (\dagger)$$

Here $[L\backslash\mathcal{S}_{\mathbf{G}}(\mathbf{L},\mathbf{M})^F/M]$ denotes a set of representatives for the cosets $L\backslash\mathcal{S}_{\mathbf{G}}(\mathbf{L},\mathbf{M})^F/M$ and $(\mathrm{ad}\, g)_M : \mathrm{Cent}(M) \to \mathrm{Cent}({}^gM)$ is the map induced by precomposing with the conjugation map $\mathrm{Inn}_{g^{-1}} : {}^gM \to M$. We say the *Mackey formula holds for* $\mathbf{G}$ if the equality ($\dagger$) holds for every pair of parabolic subgroups $\mathbf{P}$ and $\mathbf{Q}$ of $\mathbf{G}$ and every F-stable Levi complement $\mathbf{L} \subseteq \mathbf{P}$ and $\mathbf{M} \subseteq \mathbf{Q}$.

Theorem 5.34 (Deligne, Deligne–Lusztig, Bonnafé–Michel). *The Mackey formula holds for* $\mathbf{L} \subseteq \mathbf{P}$ *and* $\mathbf{M} \subseteq \mathbf{Q}$ *if one of the following conditions is satisfied:*

(i) both $\mathbf{P}$ *and* $\mathbf{Q}$ *are* F*-stable,*

(ii) either $\mathbf{L}$ *or* $\mathbf{M}$ *is a maximal torus of* $\mathbf{G}$*.*

Furthermore, the Mackey formula holds for $\mathbf{G}$ *unless* $q = 2$.

Remark 5.35. For a more precise statement of the remaining unknown cases for the Mackey formula, see [BM11].

Exercise 5.36. Assume $\mathbf{P}$ and $\mathbf{Q}$ are parabolic subgroups with common F-stable Levi complement $\mathbf{L}$. Show that if the Mackey formula holds for $\mathbf{G}$ then $R^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}} = R^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{Q}}$ and ${}^*R^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}} = {}^*R^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{Q}}$. Similarly, prove Lemma 5.5. (Hint: argue by induction on the semisimple rank of $\mathbf{G}$, which is defined to be the dimension of a maximal torus of $\mathbf{G}/R(\mathbf{G})$, and use the fact that the inner product $\langle -,-\rangle_G$ is non-degenerate.)

Exercise 5.37. Prove Lemma 5.5.

Exercise 5.38. Assume $\mathbf{B}$ and $\mathbf{B}'$ are Borel subgroups of $\mathbf{G}$ containing a common maximal torus $\mathbf{T}$. Show that $R^{\mathbf{G}}_{\mathbf{T}\subseteq\mathbf{B}} = R^{\mathbf{G}}_{\mathbf{T}\subseteq\mathbf{B}'}$ and ${}^*R^{\mathbf{G}}_{\mathbf{T}\subseteq\mathbf{B}} = {}^*R^{\mathbf{G}}_{\mathbf{T}\subseteq\mathbf{B}'}$.

◦ *Deligne–Lusztig Characters*

We will now focus on the induction map $R^{\mathbf{G}}_{\mathbf{T}\subseteq\mathbf{B}}$ where $\mathbf{T}$ is an F-stable maximal torus of $\mathbf{G}$ contained in the Borel subgroup $\mathbf{B} = \mathbf{T} \ltimes \mathbf{U}$. By the previous exercise $R^{\mathbf{G}}_{\mathbf{T}\subseteq\mathbf{B}}$ is independent of $\mathbf{B}$, hence we will simply denote this by $R^{\mathbf{G}}_{\mathbf{T}}$.

Definition 5.39. Assume $\mathbf{T}$ is an F-stable maximal torus of $\mathbf{G}$ and $\theta \in \mathrm{Irr}(T)$ then we call $R^{\mathbf{G}}_{\mathbf{T}}(\theta)$ a *Deligne–Lusztig character* of G.

Note that $R^{\mathbf{G}}_{\mathbf{T}}(\theta) \in \mathbb{Z}\,\mathrm{Irr}(G)$ is only a virtual character of G and is not necessarily a character of G. We would now like to consider a slightly better description for the virtual character $R^{\mathbf{G}}_{\mathbf{T}}(\theta)$. In general the best one can achieve is the following mild rewording.

Proposition 5.40. *For any $g \in G$ we have*

$$R^{\mathbf{G}}_{\mathbf{T}}(\theta)(g) = \frac{1}{|T|}\sum_{t\in T}\theta(t^{-1})\mathcal{L}((g,t),\mathbf{Y}^{\mathbf{G}}_{\mathbf{U}}).$$

However with this we can start to consider how much $R^{\mathbf{G}}_{\mathbf{T}}(\theta)$ depends upon the choice of $\mathbf{T}$ and θ. Before we do this we introduce the following notation.

Definition 5.41. Let $\nabla(\mathbf{G},F)$ denote the set of all pairs $(\mathbf{T},\theta)$ such that $\mathbf{T}$ is an F-stable maximal torus of $\mathbf{G}$ and $\theta \in \mathrm{Irr}(T)$. We say two pairs $(\mathbf{T}',\theta')$, $(\mathbf{T},\theta) \in \nabla(\mathbf{G},F)$ are *rationally conjugate* if there exists an element $x \in G$ such that $\mathbf{T}' = {}^x\mathbf{T}$ and $\theta' = {}^x\theta$. This defines an equivalence relation $\sim_G$ on $\nabla(\mathbf{G},F)$ and we denote the set of all equivalence classes by $\nabla(\mathbf{G},F)/G$.

Exercise 5.42. Prove that if $(\mathbf{T}',\theta') \sim_G (\mathbf{T},\theta)$ then $R^{\mathbf{G}}_{\mathbf{T}'}(\theta') = R^{\mathbf{G}}_{\mathbf{T}}(\theta)$. (Hint: Use Proposition 5.40 and (iv) from Theorem 5.27.)

Shortly we will see that the converse of Exercise 5.42 is also true. Although we cannot give a better description for $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$ in general, we can give a considerably simpler description when θ is the trivial character. For this we recall the notation concerning F-stable maximal tori developed in Remark 4.19. Given $w \in W_{\mathbf{G}}(\mathbf{T}_0)$ we define the following algebraic subset of $\mathbf{G}$ associated to w

$$\mathbf{Y}_w^{\mathbf{G}} = \mathscr{L}^{-1}(\dot{w}\mathbf{U}_0) = \{g \in \mathbf{G} \mid g^{-1}F(g) \in \dot{w}\mathbf{U}_0\},$$

where $\mathbf{U}_0 = R_u(\mathbf{B}_0)$. Let $x \in \mathscr{L}^{-1}(\dot{w}\mathbf{U}_0)$ and $t \in \widehat{T}_w$ then $xt \in \mathscr{L}^{-1}(\dot{w}\mathbf{U}_0)$ because

$$\mathscr{L}(xt) = t^{-1}\mathscr{L}(x)F(t) \in t^{-1}\dot{w}\mathbf{U}_0\dot{w}^{-1}t\dot{w} = \dot{w}(\dot{w}^{-1}t\dot{w})^{-1}\mathbf{U}_0(\dot{w}^{-1}t\dot{w}) = \dot{w}\mathbf{U}_0,$$

where the last equality follows from the fact that $\dot{w}^{-1}t\dot{w} \in \mathbf{T}_0 \leqslant N_{\mathbf{G}}(\mathbf{U}_0)$. In particular this shows that $\widehat{T}_w$ acts on $\mathbf{Y}_w^{\mathbf{G}}$ by right multiplication. To each $w \in W_{\mathbf{G}}(\mathbf{T}_0)$ we can define the variety $\mathbf{X}_w^{\mathbf{G}} = \mathscr{L}^{-1}(\dot{w}\mathbf{U}_0)/\widehat{T}_w$ to be the affine quotient by the finite group $\widehat{T}_w$. Clearly the finite group G acts on $\mathbf{X}_w^{\mathbf{G}}$ by left multiplication because for any $x \in \mathbf{X}_w$ and $g \in G$ we have $\mathscr{L}(gx) = \mathscr{L}(x)$. We then have the following description for the Deligne–Lusztig character $R_{\mathbf{T}_w}^{\mathbf{G}}(1)$.

Proposition 5.43. *For all $w \in \mathbf{W}$ we have $R_{\mathbf{T}_w}^{\mathbf{G}}(1)(g) = \mathcal{L}(g, \mathbf{X}_w^{\mathbf{G}})$ for any $g \in G$.*

Remark 5.44. We consider now whether the construction given above depends upon our choice of representative $\dot{w}$ for w. Assume $\ddot{w} \in N_{\mathbf{G}}(\mathbf{T}_0)$ is another representative for w and let $\mathbf{T}_w' = {}^{g'}\mathbf{T}_0$ for some $g' \in \mathbf{G}$ such that $\mathscr{L}(g') = \ddot{w}$. By Example 4.17 we have $\mathbf{T}_w$ and $\mathbf{T}_w'$ are conjugate under G so by Exercise 5.42 we have $R_{\mathbf{T}_w}^{\mathbf{G}}(1) = R_{\mathbf{T}_w'}^{\mathbf{G}}(1)$. In particular this says $\mathcal{L}(-, \mathbf{X}_w)$ does not depend upon our choice of $\dot{w}$.

We will now go on to state two of the most important theorems regarding Deligne–Lusztig characters, the first being a formula for computing their inner product. As a corollary of this we will see that the converse to Exercise 5.42 holds.

Theorem 5.45. *Given two pairs $(\mathbf{T}, \theta)$, $(\mathbf{T}', \theta') \in \nabla(\mathbf{G}, F)$ we have the inner product of the corresponding Deligne–Lusztig characters is*

$$\langle R_{\mathbf{T}}^{\mathbf{G}}(\theta), R_{\mathbf{T}'}^{\mathbf{G}}(\theta')\rangle = \frac{1}{|T|}|\{n \in G \mid {}^n\mathbf{T} = \mathbf{T}' \text{ and } {}^n\theta = \theta'\}|.$$

Corollary 5.46. *We have two Deligne–Lusztig characters $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$ and $R_{\mathbf{T}'}^{\mathbf{G}}(\theta')$ are orthogonal if $(\mathbf{T}, \theta) \not\sim_G (\mathbf{T}', \theta')$, in particular $R_{\mathbf{T}}^{\mathbf{G}}(\theta) = R_{\mathbf{T}'}^{\mathbf{G}}(\theta')$ if and only if $(\mathbf{T}, \theta) \sim_G (\mathbf{T}', \theta')$.*

Assume $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$ and $R_{\mathbf{T}'}^{\mathbf{G}}(\theta')$ are orthogonal then as they are virtual characters they may still have irreducible constituents in common. However if $\langle R_{\mathbf{T}}^{\mathbf{G}}(\theta), R_{\mathbf{T}}^{\mathbf{G}}(\theta)\rangle = 1$ we must have $\pm R_{\mathbf{T}}^{\mathbf{G}}(\theta)$ is an irreducible character of G. This naturally leads us to the following definition.

Definition 5.47. Let $\mathbf{T}$ be an F-stable maximal torus of $\mathbf{G}$ then we say $\theta \in \mathrm{Irr}(T)$ is in *general position* if no element of $W_{\mathbf{G}}(\mathbf{T})^F$ fixes θ.

Proposition 5.48. *If $w \in W_{\mathbf{G}}(\mathbf{T}_0)$ and $\theta \in \mathrm{Irr}(T_w)$ then the degree of $R_{\mathbf{T}_w}^{\mathbf{G}}(\theta)$ is given by $(-1)^{\ell(w)} R_{\mathbf{T}_w}^{\mathbf{G}}(\theta)(1) = [G : T_w]_{p'}$. In particular, if θ is in general position then $(-1)^{\ell(w)} R_{\mathbf{T}_w}^{\mathbf{G}}(\theta) \in \mathrm{Irr}(G)$.*

By $[G : T_w]$ we mean the finite group index of T_w in G. The subscript p' is to denote the largest divisor of the index which is coprime to p. We now wish to give our second important theorem regarding Deligne–Lusztig characters. However, before we can do this we need to recast our labelling set for Deligne–Lusztig characters in terms of the dual group.

Definition 5.49. Let $\nabla^\star(\mathbf{G}, F)$ denote the set of all pairs $(\mathbf{T}^\star, s)$ such that $\mathbf{T}^\star$ is an $F^\star$-stable maximal torus of $\mathbf{G}^\star$ and $s \in T^\star$. We say two pairs $(\mathbf{T}^{\star\prime}, s')$, $(\mathbf{T}^\star, s) \in \nabla^\star(\mathbf{G}, F)$ are *rationally conjugate* if there exists an element $x \in G^\star$ such that $\mathbf{T}^{\star\prime} = {}^x\mathbf{T}^\star$ and $s' = {}^x s$. This defines an equivalence relation $\sim_{G^\star}$ on $\nabla^\star(\mathbf{G}, F)$ and we denote the set of all equivalence classes by $\nabla^\star(\mathbf{G}, F)/G^\star$.

There is a strong relationship between the sets of equivalence classes $\nabla^\star(\mathbf{G}, F)/G^\star$ and $\nabla(\mathbf{G}, F)/G$. Recalling the notion of a dual maximal torus from Definition 4.34, we have the following.

Lemma 5.50. *We have a well-defined bijective correspondence*

$$\nabla(\mathbf{G}, F)/G \to \nabla^\star(\mathbf{G}, F)/G^\star$$

such that $(\mathbf{T}, 1) \mapsto (\mathbf{T}^\star, 1)$, where 1 denotes the trivial character of T or the identity in $G^\star$.

Assume $(\mathbf{T}, \theta) \in \nabla(\mathbf{G}, F)$ corresponds to $(\mathbf{T}^\star, s)$ under the bijection in Lemma 5.50 then we may write $R_{\mathbf{T}^\star}^{\mathbf{G}}(s)$ for $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$ without ambiguity. We can now state the second main result for Deligne–Lusztig characters.

Theorem 5.51. *Suppose $(\mathbf{T}^{\star\prime}, s')$, $(\mathbf{T}^\star, s) \in \nabla^\star(\mathbf{G}, F)$ are two pairs such that s' and s are not conjugate in $G^\star$ then the corresponding Deligne–Lusztig characters $R_{\mathbf{T}^{\star\prime}}^{\mathbf{G}}(s')$ and $R_{\mathbf{T}^\star}^{\mathbf{G}}(s)$ have no irreducible constituent in common.*

In particular, Theorem 5.51 and Proposition 5.48 tell us the following. Let $\{w_1, \ldots, w_r\}$ be a set of representatives for the distinct F-conjugacy classes in $H^1(F, \mathbf{W})$. For each maximal torus $\mathbf{T}_{w_i}$ and irreducible character $\theta \in \operatorname{Irr}(T_{w_i})$ in general position we have $(-1)^{\ell(w_i)} R_{\mathbf{T}_{w_i}}^{\mathbf{G}}(\theta)$ will be an irreducible character of G and for distinct $1 \leqslant i, j \leqslant r$ we will have $(-1)^{\ell(w_i)} R_{\mathbf{T}_{w_i}}^{\mathbf{G}}(\theta) \neq (-1)^{\ell(w_j)} R_{\mathbf{T}_{w_j}}^{\mathbf{G}}(\theta)$.

∘ *Lusztig series and the Jordan decomposition of characters*

Previously we observed that Deligne–Lusztig characters furnish us with many irreducible characters of our finite reductive group G. However, it is not the case that every class function of G can be written in terms of Deligne–Lusztig characters. More precisely, we have the following. Let

$$\operatorname{Cent}_{\mathrm{DL}}(G) = \operatorname{span}_{\overline{\mathbb{Q}}_\ell}\{R_{\mathbf{T}}^{\mathbf{G}}(\theta) \mid (\mathbf{T}, \theta) \in \nabla(\mathbf{G}, F)\}$$

be the subspace of $\operatorname{Cent}(G)$ spanned by Deligne–Lusztig characters then it can happen that $\operatorname{Cent}_{\mathrm{DL}}(G) \neq \operatorname{Cent}(G)$. We call the elements of $\operatorname{Cent}_{\mathrm{DL}}(G)$ *uniform class functions*. Although this subspace is usually proper, several important class functions are always uniform.

Lemma 5.52. *The regular character of G is a uniform class function. In particular, as every irreducible character of G occurs in the regular character we must have every irreducible character of G occurs in some Deligne–Lusztig character.*

We now use the Deligne–Lusztig characters to give an initial partitioning of the irreducible characters of G. Using Theorem 5.51, we have two Deligne–Lusztig characters $R_{\mathbf{T}^{\star\prime}}^{\mathbf{G}}(s')$ and $R_{\mathbf{T}^\star}^{\mathbf{G}}(s)$ have no irreducible constituents in common unless s', s are in the same $G^\star$-conjugacy class.

Definition 5.53. Let $[s] = [s]_{G^\star}$ be the $G^\star$-conjugacy class of semisimple elements containing $s \in G^\star$. We define the *Lusztig series* of G associated to $[s]$ to be the set

$$\mathcal{E}(G, s) = \mathcal{E}(G, [s]) = \{\chi \in \operatorname{Irr}(G) \mid \langle \chi, R_{\mathbf{T}^\star}^{\mathbf{G}}(s)\rangle \neq 0 \text{ for some } (\mathbf{T}^\star, s) \in \nabla^\star(\mathbf{G}, F)\}.$$

Remark 5.54. We have chosen here to define the Lusztig series to be the so called *rational Lusztig series*. However it should be noted that many people would define a Lusztig series to be the *geometric Lusztig series*.

By Lemma 5.52 every irreducible character of G occurs in some Deligne–Lusztig character. Therefore by Theorem 5.51 the Lusztig series give a partition of the irreducible characters of G

$$\operatorname{Irr}(G) = \bigsqcup_{[s]} \mathcal{E}(G, s),$$

where $[s]$ runs over all $G^\star$-conjugacy classes of semisimple elements.

Definition 5.55. If $\chi \in \mathcal{E}(G,1)$ then we call χ a *unipotent character* of G.

Example 5.56. If G is $\mathrm{GL}_n(q)$ then $\mathcal{E}(G,1)$ is simply the irreducible constituents of the Harish-Chandra induced character $R_{\mathbf{T}_0}^{\mathbf{G}}(1)$. In particular, $\mathcal{E}(G,1)$ is parameterised by the irreducible characters of the symmetric group $\mathfrak{S}_n$.

One of the crowning achievements of Lusztig was to relate the elements in the Lusztig series $\mathcal{E}(G,s)$ to a set of unipotent characters. This result is typically refereed to as the Jordan decomposition of characters and shows the prominant role played by the dual group.

Theorem 5.57 (Jordan decomposition of characters). *Let* $\mathbf{G}$ *be any connected reductive algebraic group and* $s \in G^\star$ *a semisimple element such that the centraliser* $C_{\mathbf{G}^\star}(s)$ *is connected. Then* $C_{\mathbf{G}^\star}(s)$ *is a connected reductive algebraic group with Frobenius endomorphism* $F^\star$ *and we have a bijection*

$$\mathcal{E}(G,s) \to \mathcal{E}(C_{G^\star}(s),1),$$

denoted $\chi \mapsto \chi_u$, *such that*

$$\langle \chi, R_{\mathbf{T}^\star}^{\mathbf{G}}(s)\rangle_G = \pm\langle \chi_u, R_{\mathbf{T}^\star}^{C_{\mathbf{G}^\star}(s)}(1)\rangle_{C_{G^\star}(s)}$$

for all $(\mathbf{T}^\star, s) \in \nabla^\star(\mathbf{G}, F)$

Remark 5.58. We will not define this precisely here but merely remark that the sign $\pm$ can be described explicitly using the notion of $\mathbb{F}_q$-rank.

The condition that the centraliser $C_{\mathbf{G}^\star}(s)$ is connected is crucial for this result to hold. If this assumption is dropped then the theory we have developed here no longer applies and even the definition of the set $\mathcal{E}(C_{G^\star}(s),1)$ does not make sense. However, using Clifford theory, Lusztig has extended this result to the case where $C_{\mathbf{G}^\star}(s)$ is disconnected but we will not discuss this here. We finish with a condition that ensures $C_{\mathbf{G}^\star}(s)$ is always connected.

Proposition 5.59. *Assume the centre* $Z(\mathbf{G})$ *of* $\mathbf{G}$ *is connected then* $C_{\mathbf{G}^\star}(s)$ *is connected for every semisimple element* $s \in \mathbf{G}^\star$.

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