Exercises

Exercise 12.1. Describe all closed $\mathrm{GL}$-subvarieties of $\mathbb{A}^{(1)} \times \mathbb{A}^{(1)}$.

Exercise 12.2. Show that the rank $\leq s$ loci account for all the non-empty proper closed $\mathrm{GL}$-subvarieties of $\mathbb{A}^{(2)}$.

Example 12.7. Let $\lambda = [(d_1), \ldots, (d_r)]$. Show that any map of $\mathrm{GL}$-varieties $\varphi: \mathbb{A}^{\lambda} \to \mathbb{A}^{(e)}$ has the form

$$\varphi(f_1, \ldots, f_r) = \Phi(f_1, \ldots, f_r)$$

where $\Phi \in k[T_1, \ldots, T_r]$ is polynomial. Moreover, show that $\Phi$ is homogeneous of degree $e$ if $T_i$ is given degree $d_i$. (A map of $\mathrm{GL}$-varieties is simply a $\mathrm{GL}$-equivariant map of schemes over $k$.)
Additional exercises

Exercise 12.3. A point in a \(\text{GL}\) variety is \(\text{GL}\)-generic if it has dense \(\text{GL}\) orbit.

(a) Show that any element of \(A^{(2)}\) of infinite strength is \(\text{GL}\)-generic. In fact, this is true in \(A^{(d)}\) as well, but the proof is much harder.

(b) Let \(f = \sum_{i \geq 0} x_{3i+1} x_{3i+2} x_{3i+3}\), regarded as a point of \(A^{(3)}\). Show that \(f\) is \(\text{GL}\)-generic. (Hint: show that any cubic in \(n\) variables can be obtained as a limit of points in the \(\text{GL}\) orbit of \(f\).)

(c) On the other hand, show that there is no point in \(\text{Sym}^3(k^n)\) with dense \(\text{GL}_n\)-orbit. Thus the above result is somewhat surprising.

(d) Show that if \(\lambda\) is any tuple of non-empty partitions the space \(A^\lambda\) admits a \(\text{GL}\)-generic point.

Exercise 12.4. Let \(X = \text{Spec}(R)\) be an irreducible \(\text{GL}\) variety. The invariant function field of \(X\) is the field \(k(X)_{\text{GL}}\) of \(\text{GL}\)-invariant elements in the function field \(k(X) = \text{Frac}(R)\). Compute this when \(X\) is the closed subvariety of \(A^{(1)} \times A^{(1)}\) consisting of pairs \((x, y)\) that are linearly dependent.