

Lecture 2: main exercises

Exercise 2.1. Let I be a set equipped with a non-principal ultrafilter \mathcal{F} . Let $\{K_i\}_{i \in I}$ be a family of fields, and let K^* be their ultraproduct.

- (a) Show that K^* is a field under coordinatewise operations. Pay special attention to reciprocals.
- (b) Suppose that I is the set of prime numbers and $K_p = \mathbf{F}_p$. Show that K^* has characteristic 0.

Now suppose that for each $i \in I$ we have a K_i -vector space V_i . Let V^* be the ultraproduct of the V_i 's.

- (c) Show that V^* is naturally a K^* -vector space.
- (d) Suppose $\dim_{K_i}(V_i) = n$ for all $i \in I$. Show that $\dim_{K^*}(V^*) = n$.

Exercise 2.2. Let I be a set equipped with a non-principal ultrafilter \mathcal{F} . Let \mathbf{R}^* be the ultrapower of the real numbers \mathbf{R} . (Ultrapower is just the case where the X_i 's are all the same set.) Identify $a \in \mathbf{R}$ with the element (a, a, \dots) in \mathbf{R}^* .

- (a) Show that \mathbf{R}^* carries a natural total order.
- (b) Show that there is an element $\epsilon \in \mathbf{R}^*$ such that $0 < \epsilon < a$ for all positive $a \in \mathbf{R}$.

An element ϵ as in (b) is called an *infinitesimal*. These elements form the basis of non-standard analysis.

Lecture 2: additional exercises

Exercise 2.3. Show that any ultrafilter on a finite set is principal.

Exercise 2.4. Show that a non-principal ultrafilter contains all cofinite sets.

Exercise 2.5. Let I be the set of prime numbers, for $p \in I$ let $K_p = \mathbf{F}_p$, and let K^* be the ultraproduct of the K_p 's with respect to an ultrafilter \mathcal{F} on I .

- (a) Show that one can choose \mathcal{F} so that -1 is a square in K^* .
- (b) Show that one can choose \mathcal{F} so that -1 is not a square in K^* .
- (c) Try to figure out various possibilities for the behavior of algebraic numbers in K^* . E.g., is it possible for K^* to contain no irrational algebraic numbers?

Exercise 2.6. Prove the correspondence between ultrafilters and maximal ideals stated in lecture.