Lecture 3: main exercises

Exercise 3.1. Work over the complex numbers, and let d be a positive integer. Show that $\sum_{i \geq 1} x_i^d$ has infinite strength in $\mathbf{R}$. [Hint: show that if this element had strength $s < \infty$ then any degree d polynomial in n variables would have strength $\leq s$ and obtain a contradiction.]

Exercise 3.2. Let $f_i \in k[x_1, x_2, \ldots]$ be homogeneous of degree d and let $f \in \mathbf{S}$ be the corresponding element. Show that f has finite strength if and only if there is some set J in the ultrafilter such that the elements $\{f_j\}_{j \in J}$ have bounded strength. Formulate and prove a similar result for collective strength of a finite collection of elements.

Lecture 3: additional exercises

Exercise 3.3. Suppose that k is a finite field. For each $n \geq 1$, let f_n be a homogeneous degree d polynomial in $k[x_1, \ldots, x_n]$.

(a) Show that there is a homogeneous degree d element f in $\mathbf{R}$ such that f maps to f_n for infinitely many n. [Hint: $\mathbf{R}_d$ is a compact topological space (how?).]
(b) Suppose that the strength of the f_n's goes to infinity. Show that f has infinite strength.
(c) Suppose that the f_n's have bounded strength. Show that f has finite strength.

Exercise 3.4. Suppose $\mathrm{char}(k) \neq 2$. Let $f = \sum_{1 \leq i \leq j} a_{i,j} x_i x_j$ be a degree two element of $\mathbf{R}$.

(a) Show that f has strength 1 if and only if $a_{i,j} a_{k,\ell} = a_{i,\ell} a_{j,k}$ (with the convention $a_{i,j} = a_{j,i}$).
(b) Show that the condition "f has strength $\leq s$" is equivalent to a system of polynomial equations in the coefficients $a_{i,j}$.

Exercise 3.5. Let $\mathbf{R}_{k^*}$ be the graded inverse limit of the rings $k^*[x_1, \ldots, x_n]$.

(a) Construct a natrual homomorphism of k^*-algebras $\varphi \colon \mathbf{S} \to \mathbf{R}_{k^*}$. [Hint: given $f \in \mathbf{S}$ define $\varphi(f)$ by specifying the coefficient of each monomial.]
(b) Determine if φ is injective or surjective (or both or neither).

Exercise 3.6. Write down an element of the inverse limit of the R_n's in the category of (ungraded) rings that does not belong to $\mathbf{R}$.