Lecture 4: main exercises

Exercise 4.1. Show that the ultraproduct ring $\mathbf{S}$ has enough derivations, and is therefore a polynomial ring.

Exercise 4.2. Show that the equivalence "polynomial ring if and only if enough derivations" fails in both directions in positive characteristic. [Hint: show that $k[x]/(x^p)$ has enough derivations when k has characteristic p.]

Lecture 4: additional exercises

Exercise 4.3. Let R be a k-algebra. A *Hasse derivation* on R is a sequence $\{\partial_i\}_{i\geq 0}$ such that each ∂_i is k-linear and $\partial_n(xy) = \sum_{i+j=n} \partial_i(x)\partial_j(y)$. The intuition is that ∂_i should be like $\frac{1}{i!}\partial_1^i$. Construct a Hasse derivation on $k[x]$ (do not assume k has characteristic 0).

Exercise 4.4. Suppose k has characteristic $p > 0$ and is perfect (every element is a pth power). We say that R has *enough Hasse derivations* if whenever f is a homogeneous element that is not a pth power there is a homogeneous Hasse derivation ∂ of negative degree such that $\partial_1(f) \neq 0$. We have the following theorem [ESS2, Theorem 2.11]: a graded k-algebra is a polynomial ring if and only if it has enough Hasse derivations. The proof is a bit involved, so we'll just look at some special cases and adjacent results:

(a) Show that a polynomial ring has enough Hasse derivations.
(b) Show that if R has enough Hasse derivations then R is reduced. (This is the first step in the proof of the theorem.)
(c) Using the theorem, show that $\mathbf{R}$ is a polynomial ring.

Exercise 4.5. Let A_n be the exterior algebra on an n-dimensional vector space, regarded as a graded algebra, and let $\mathbf{A}$ be the inverse limit of the A_n's in the category of graded algebras.

(a) Show that $\mathbf{A}$ is *not* an exterior algebra on some vector space.
(b) Presumably, $\mathbf{A}$ should be a free algebra of a certain kind; can you figure out which kind?
(c) Prove your conjecture in (b). (As far as I know, this is still an open problem, and could make a nice little paper.)

Exercise 4.6. Prove "polynomial ring if and only if enough derivations" (in characteristic 0) without the assumption that R is finitely generated. (The argument needs only minor modifications.)