Exercise 6.1. Let \( f_1, \ldots, f_r \) and \( g_1, \ldots, g_s \) be sequences of homogeneous elements of \( R_\infty \) indexed by \( I \), and let \( f_1, \ldots, f_r \) and \( g_1, \ldots, g_s \) be the corresponding elements of \( S \). Define

\[
\begin{align*}
\mathbf{a}_i &= (f_{1,i}, \ldots, f_{r,i}) \\
\mathbf{b}_i &= (g_{1,i}, \ldots, g_{s,i}) \\
\mathbf{a} &= (f_1, \ldots, f_r) \\
\mathbf{b} &= (g_1, \ldots, g_s).
\end{align*}
\]

Show that \( \mathbf{a} = \mathbf{b} \) if and only if \( \mathbf{a}_i = \mathbf{b}_i \) for all \( i \) belonging to some set in the ultrafilter.

Exercise 6.2. Let \( \mathbf{a} \) be a finitely generated ideal of \( S \) and let \( \mathbf{a}_\star \) be the corresponding sequence of ideals in \( R_\infty \). Show that \( \mathbf{a} \) is radical if and only if \( \mathbf{a}_i \) is radical for all \( i \) in some set in the ultrafilter. To do this, make use of the following result:

\( (*) \) Given degrees \( d_1, \ldots, d_r \) there exists a positive integer \( M = M(d_1, \ldots, d_r) \) with the following property: if \( \mathbf{a} \) is an ideal in a polynomial ring generated by \( r \) homogeneous elements of degrees \( d_1, \ldots, d_r \) then \( \text{rad}(\mathbf{a})^M \subseteq \mathbf{a} \).

This statement is a consequence of the effective Nullstellensatz.

Exercise 6.3. Prove a version of the main theorem (about codimension in \( S \)) for the inverse limit ring \( \mathbf{R} \).