Exercise 7.1. Compute the weights of $\mathrm{Sym}^2(\mathbf{C}^3)$ as a representation of $\mathbf{GL}_3$, and draw the Hasse diagram of the dominance order.

Exercise 7.2. Compute the multiset of weights of $(\mathbf{C}^2)^{\otimes 4}$ as a representation of $\mathbf{GL}_2$. Decompose this representation into irreducible representations.

Exercise 7.3. This exercise walks you through the decomposition of the 27-dimensional representation $V = (\mathbf{C}^3)^{\otimes 3}$ of $\mathbf{GL}_3$.

(a) Compute the multiset of weights of $\bigwedge^2(\mathbf{C}^3)$, and then the multiset of weights of $W = \bigwedge^2(\mathbf{C}^3) \otimes \mathbf{C}^3$. Deduce that $W = L_{(2,1,0)} \oplus \bigwedge^3(\mathbf{C}^3)^{\oplus m}$ for some unknown multiplicity m. If you know m, how do you compute the multiset of weights of $L_{(2,1,0)}$?

(b) With what multiplicity does $\bigwedge^3(\mathbf{C}^3)$ appear as a direct summand of V? [Hint: This one-dimensional representation can only appear as a summand of a particular six-dimensional weight space. The symmetric group $S_3 \subset \mathbf{GL}_3$ acts in a particular way on this six-dimensional space, and on $\bigwedge^3(\mathbf{C}^3)$.]

(c) Use part (b) to compute m and the weights of $L_{(2,1,0)}$.

(d) Now compute the decomposition of $\mathrm{Sym}^2(\mathbf{C}^3) \otimes \mathbf{C}^3$ into irreducible representations.

(e) Now compute the decomposition of V into irreducible representations. [Hint: First decompose $\mathbf{C}^3 \otimes \mathbf{C}^3$, then tensor with $\mathbf{C}^3$.]

(f) (Come back after Lecture 8) Match your answer with the answer that Schur–Weyl duality gives.

Exercise 7.4. If G acts on V, then G acts on $V^* = \{f\colon V \to \mathbf{C}\}$ by the formula $g \cdot f(v) = f(g^{-1}v)$.

(a) For $G = \mathbf{GL}_2$ and $V = \mathbf{C}^2$, write down the matrix for the action of $g \in G$ on V^*, with respect to the dual standard basis.
(b) From this you might be able to guess the general formula which takes the matrix of g on a basis of V and produces the matrix of g on the dual basis of V^*. Prove that $(\mathbf{C}^n)^* \otimes \det$ is a polynomial representation of $\mathbf{GL}_n$, whereas $(\mathbf{C}^n)^*$ is only rational (for $n > 1$).

Exercise 7.5. Prove the classification of irreducible rational representations of $\mathbf{GL}_1$.

Exercise 7.6. Let $\rho\colon \mathbf{GL}_1 \to \mathbf{GL}_2$ be a homomorphism given by

$$\rho(z) = \begin{pmatrix} z^k & r(z) \\ 0 & z^k \end{pmatrix}$$

for some $k \in \mathbf{Z}$ and some rational function $r(z)$. Deduce that $r = 0$. This does not prove the semisimplicity theorem for $\mathbf{GL}_1$, but it certainly makes it believable. [Hint: Compare certain coefficients in $\rho(z)^2 = \rho(z^2)$.]

Exercise 7.7. Let $Z \subset \mathbf{GL}_n$ be the group of scalar matrices; this is isomorphic to $\mathbf{GL}_1$. Let V be a rational representation of $\mathbf{GL}_n$ and let $V = \bigoplus_{k\in\mathbf{Z}} V_k$ be the weight decomposition for Z. Show that each V_k is a $\mathbf{GL}$-subrepresentation of V. We thus see that every rational $\mathbf{GL}$-representation admits a canonical $\mathbf{Z}$-grading.

Exercise 7.8. Let V be a rational representation of $\mathbf{GL}_n$.

(a) Let $\langle , \rangle_0$ be a Hermitian form on V. Define a new Hermitian form $\langle , \rangle$ on V by the formula
$$\langle v, w\rangle = \int_{\mathbf{U}_n} \langle gv, gw\rangle_0 dg.$$
Here $\mathbf{U}_n \subset \mathbf{GL}_n$ is the unitary group and dg is its Haar measure. Show that $\langle , \rangle$ is $\mathbf{GL}_n$-invariant, i.e., $\langle gv, gw\rangle = \langle v, w\rangle$ for $g \in \mathbf{GL}_n$ and $v, w \in V$. [Hint: $\mathbf{U}_n$ is Zariski dense in $\mathbf{GL}_n$.]
(b) Let W be a $\mathbf{GL}_n$-subrepresentation of V and let W' be its orthogonal complement under $\langle , \rangle$. Show that W' is also a $\mathbf{GL}_n$-subrepresentation and that $V = W \oplus W'$.
(c) Show that V decomposes as a direct sum of irreducible representations (Theorem **??**).