Exercise 7.1. Compute the weights of $\text{Sym}^2(C^3)$ as a representation of $GL_3$, and draw the Hasse diagram of the dominance order.

Exercise 7.2. Compute the multiset of weights of $(C^2)^\otimes 4$ as a representation of $GL_2$. Decompose this representation into irreducible representations.

Exercise 7.3. This exercise walks you through the decomposition of the 27-dimensional representation $V = (C^3)^\otimes 3$ of $GL_3$.

(a) Compute the multiset of weights of $\Lambda^2(C^3)$, and then the multiset of weights of $W = \Lambda^2(C^3) \otimes C^3$. Deduce that $W = L_{(2,1,0)} \oplus \Lambda^3(C^3)^{\pm m}$ for some unknown multiplicity $m$. If you know $m$, how do you compute the multiset of weights of $L_{(2,1,0)}$?

(b) With what multiplicity does $\Lambda^3(C^3)$ appear as a direct summand of $V$? [Hint: This one-dimensional representation can only appear as a summand of a particular six-dimensional weight space. The symmetric group $S_3 \subset GL_3$ acts in a particular way on this six-dimensional space, and on $\Lambda^3(C^3)$.

(c) Use part (b) to compute $m$ and the weights of $L_{(2,1,0)}$.

(d) Now compute the decomposition of $\text{Sym}^2(C^3) \otimes C^3$ into irreducible representations.

(e) Now compute the decomposition of $V$ into irreducible representations. [Hint: First decompose $C^3 \otimes C^3$, then tensor with $C^3$.

(f) (Come back after Lecture 8) Match your answer with the answer that Schur–Weyl duality gives.
Exercise 7.4. If $G$ acts on $V$, then $G$ acts on $V^* = \{ f : V \to \mathbb{C} \}$ by the formula $g \cdot f(v) = f(g^{-1}v)$.

(a) For $G = \text{GL}_2$ and $V = \mathbb{C}^2$, write down the matrix for the action of $g \in G$ on $V^*$, with respect to the dual standard basis.

(b) From this you might be able to guess the general formula which takes the matrix of $g$ on a basis of $V$ and produces the matrix of $g$ on the dual basis of $V^*$. Prove that $(\mathbb{C}^n)^* \otimes \text{det}$ is a polynomial representation of $\text{GL}_n$, whereas $(\mathbb{C}^n)^*$ is only rational (for $n > 1$).

Exercise 7.5. Prove the classification of irreducible rational representations of $\text{GL}_1$.

Exercise 7.6. Let $\rho : \text{GL}_1 \to \text{GL}_2$ be a homomorphism given by

$$\rho(z) = \begin{pmatrix} z^k & r(z) \\ 0 & z^k \end{pmatrix}$$

for some $k \in \mathbb{Z}$ and some rational function $r(z)$. Deduce that $r = 0$. This does not prove the semisimplicity theorem for $\text{GL}_1$, but it certainly makes it believable. [Hint: Compare certain coefficients in $\rho(z)^2 = \rho(z^2)$.]

Exercise 7.7. Let $Z \subset \text{GL}_n$ be the group of scalar matrices; this is isomorphic to $\text{GL}_1$. Let $V$ be a rational representation of $\text{GL}_n$ and let $V = \bigoplus_{k \in \mathbb{Z}} V_k$ be the weight decomposition for $Z$. Show that each $V_k$ is a $\text{GL}$-subrepresentation of $V$. We thus see that every rational $\text{GL}$-representation admits a canonical $\mathbb{Z}$-grading.

Exercise 7.8. Let $V$ be a rational representation of $\text{GL}_n$.

(a) Let $\langle , \rangle_0$ be a Hermitian form on $V$. Define a new Hermitian form $\langle , \rangle$ on $V$ by the formula

$$\langle v, w \rangle = \int_{U_n} \langle gv, gw \rangle_0 dg.$$

Here $U_n \subset \text{GL}_n$ is the unitary group and $dg$ is its Haar measure. Show that $\langle , \rangle$ is $\text{GL}_n$-invariant, i.e., $\langle gv, gw \rangle = \langle v, w \rangle$ for $g \in \text{GL}_n$ and $v, w \in V$. [Hint: $U_n$ is Zariski dense in $\text{GL}_n$.]

(b) Let $W$ be a $\text{GL}_n$-subrepresentation of $V$ and let $W'$ be its orthogonal complement under $\langle , \rangle$. Show that $W'$ is also a $\text{GL}_n$-subrepresentation and that $V = W \oplus W'$.

(c) Show that $V$ decomposes as a direct sum of irreducible representations (Theorem ??).