Lecture 8: main exercises

Exercise 8.1. Put $T_k(V) = V^{\otimes k}$. Show that the following conditions on a functor F are equivalent:

(a) F is polynomial.
(b) F is a subfunctor of a direct sum of T_k's.
(c) F is a direct summand of a direct sum of T_k's.

As an application, show that a tensor product of polynomial functors is a polynomial functor.

Exercise 8.2. Let V be a vector space. Show that $\mathbf{S}_{(n-1,1)}(V)$ is naturally isomorphic to the kernel of the multiplication map $\mathrm{Sym}^{n-1}(V) \otimes V \to \mathrm{Sym}^n(V)$.

Exercise 8.3. Let F be a polynomial functor. Define $\ell(F)$ to be the supremum of the set

$$\{\ell(\lambda) \mid \mathbf{S}_\lambda \text{ is a summand of } F\}$$

Suppose that $\ell(F) = n$ is finite (we say that F is *bounded*). Show that the function

$$\begin{gathered}\{\text{subfunctors of } F\} \to \{\mathbf{GL}_n\text{-subrepresentations of } F(\mathbf{C}^n)\} \\ G \mapsto G(\mathbf{C}^n)\end{gathered}$$

is a well-defined bijection. (This gives a precise sense in which evaluating on $\mathbf{C}^n$ does not lose information.)

Exercise 8.4. Let V and W be vector spaces. Show that there are natural isomorphisms

$$\mathrm{Sym}^n(V \oplus W) = \bigoplus_{i+j=n} \mathrm{Sym}^i(V) \otimes \mathrm{Sym}^j(W)$$
$$\mathrm{Sym}^n(V \otimes W) = \bigoplus_{\lambda \vdash n} \mathbf{S}_\lambda(V) \otimes \mathbf{S}_\lambda(W)$$

These are known as the *binomial theorem* and *Cauchy identity*. [Hint: for the Cauchy identity, decompose $V^{\otimes n}$ and $W^{\otimes n}$ using Schur–Weyl duality, then tensor these together and take $\mathfrak{S}_n$-invariants. Second hint: irreducible representations of $\mathfrak{S}_n$ are self-dual.]

Exercise 8.5. In what follows, V denotes a vector space.

(a) Let $S(V) = \mathrm{Sym}(\mathbf{C}^2 \otimes V)$. Show that S is a polynomial functor.
(b) Show that S is bounded, in the sense of Exercise 8.3.
(c) An *ideal* of S is a subfunctor $\mathfrak{a}$ such that $\mathfrak{a}(V)$ is an ideal of $S(V)$ for all V. Show that ideals of S satisfy the ascending chain condition. [Hint: use Exercise 8.3.]

Exercise 8.6. Look at the six-dimensional $(1,1,1)$ weight space inside $(\mathbf{C}^3)^{\otimes 3}$. This has an action of $\mathfrak{S}_3$ by permuting the tensor factors, and a separate action of $\mathfrak{S}_3 \subset \mathbf{GL}_3$. Using either action to identify this weight space with the regular representation of $\mathfrak{S}_3$ on $\mathbf{C}[\mathfrak{S}_3]$, identify the other action.

Exercise 8.7. Consider the Specht representation of $\mathfrak{S}_4$ for the partition $\lambda = (2,2)$, living inside the polynomial ring $\mathbf{C}[x_1, x_2, x_3, x_4]$.

(a) Identify the basis of polynomials associated to standard Young tableaux.
(b) There's a non-standard tableau whose polynomial is $(x_1 - x_4)(x_2 - x_3)$. Express this polynomial in terms of the basis. You may also want to practice by expressing the polynomials associated to other non-standard tableaux in terms of the basis.
(c) Prove that this Specht representation is irreducible.
(d) Let $s = (12) \in \mathfrak{S}_4$. Find an eigenbasis for s. Show that this is a simultaneous eigenbasis for the four *Young-Jucys-Murphy operators*

$$j_1 = 0, \qquad j_2 = (12), \qquad j_3 = (13) + (23), \qquad j_4 = (14) + (24) + (34)$$

in $\mathbf{C}[\mathfrak{S}_4]$. Find the eigenvalues of these YJM operators, and try to relate them to the standard young tableaux of shape λ.

(e) The Specht representation generates an $\mathfrak{S}_4$-invariant ideal $I_\lambda \subset \mathbf{C}[x_1, x_2, x_3, x_4]$, whose vanishing set is an $\mathfrak{S}_4$-invariant algebraic set inside $\mathbf{C}^4$. Describe this vanishing set.

Exercise 8.8. Compute all irreducible representations of $\mathfrak{S}_4$ and their dimensions. For $n \in \{2, 3, 4\}$, write down the decomposition of $(\mathbf{C}^n)^{\otimes 4}$ given by Schur–Weyl duality. Compute the dimensions of L_λ for each summand in this decomposition, and make sure the overall dimensions add up appropriately.

Exercise 8.9. Let V be a rational $\mathbf{GL}_n$ representation. Prove that $\mathrm{End}_{\mathbf{GL}_n}(V)$ is semisimple. Prove that the multiplicity spaces in V are irreducible modules for $\mathrm{End}_{\mathbf{GL}_n}(V)$. [Hint: What are the irreducible representations of a matrix algebra? Of a product of matrix algebras?]