Exercise 8.1. Put $T_k(V) = V^{\otimes k}$. Show that the following conditions on a functor $F$ are equivalent:

(a) $F$ is polynomial.
(b) $F$ is a subfunctor of a direct sum of $T_k$’s.
(c) $F$ is a direct summand of a direct sum of $T_k$’s.

As an application, show that a tensor product of polynomial functors is a polynomial functor.

Exercise 8.2. Let $V$ be a vector space. Show that $S_{(n-1,1)}(V)$ is naturally isomorphic to the kernel of the multiplication map $\text{Sym}^{n-1}(V) \otimes V \rightarrow \text{Sym}^n(V)$.

Exercise 8.3. Let $F$ be a polynomial functor. Define $\ell(F)$ to be the supremum of the set

$$\{\ell(\lambda) \mid S_\lambda \text{ is a summand of } F\}$$

Suppose that $\ell(F) = n$ is finite (we say that $F$ is bounded). Show that the function

$$\{\text{subfunctors of } F\} \rightarrow \{\text{GL}_n\text{-subrepresentations of } F(\mathbb{C}^n)\}$$

$$G \mapsto G(\mathbb{C}^n)$$

is a well-defined bijection. (This gives a precise sense in which evaluating on $\mathbb{C}^n$ does not lose information.)
Lecture 8: additional exercises

Exercise 8.4. Let \( V \) and \( W \) be vector spaces. Show that there are natural isomorphisms

\[
\text{Sym}^n(V \oplus W) = \bigoplus_{i+j=n} \text{Sym}^i(V) \otimes \text{Sym}^j(W)
\]

\[
\text{Sym}^n(V \otimes W) = \bigoplus_{\lambda \vdash n} S_\lambda(V) \otimes S_\lambda(W)
\]

These are known as the binomial theorem and Cauchy identity. [Hint: for the Cauchy identity, decompose \( V \otimes^n \) and \( W \otimes^n \) using Schur–Weyl duality, then tensor these together and take \( S_n \)-invariants. Second hint: irreducible representations of \( S_n \) are self-dual.]

Exercise 8.5. In what follows, \( V \) denotes a vector space.

(a) Let \( S(V) = \text{Sym}(C^2 \otimes V) \). Show that \( S \) is a polynomial functor.

(b) Show that \( S \) is bounded, in the sense of Exercise 8.3.

(c) An ideal of \( S \) is a subfunctor \( \alpha \) such that \( \alpha(V) \) is an ideal of \( S(V) \) for all \( V \). Show that ideals of \( S \) satisfy the ascending chain condition. [Hint: use Exercise 8.3.]

Exercise 8.6. Look at the six-dimensional \((1, 1, 1)\) weight space inside \((C^3)^{\otimes 3}\). This has an action of \( \mathfrak{S}_3 \) by permuting the tensor factors, and a separate action of \( \mathfrak{S}_3 \subset \text{GL}_3 \). Using either action to identify this weight space with the regular representation of \( \mathfrak{S}_3 \) on \( C[\mathfrak{S}_3] \), identify the other action.

Exercise 8.7. Consider the Specht representation of \( \mathfrak{S}_4 \) for the partition \( \lambda = (2, 2) \), living inside the polynomial ring \( C[x_1, x_2, x_3, x_4] \).

(a) Identify the basis of polynomials associated to standard Young tableaux.

(b) There’s a non-standard tableau whose polynomial is \((x_1-x_4)(x_2-x_3)\). Express this polynomial in terms of the basis. You may also want to practice by expressing the polynomials associated to other non-standard tableaux in terms of the basis.

(c) Prove that this Specht representation is irreducible.

(d) Let \( s = (12) \in \mathfrak{S}_4 \). Find an eigenbasis for \( s \). Show that this is a simultaneous eigenbasis for the four Young–Jucys–Murphy operators

\[
j_1 = 0, \quad j_2 = (12), \quad j_3 = (13) + (23), \quad j_4 = (14) + (24) + (34)
\]

in \( C[\mathfrak{S}_4] \). Find the eigenvalues of these YJM operators, and try to relate them to the standard young tableaux of shape \( \lambda \).
(e) The Specht representation generates an \( S_4 \)-invariant ideal \( I_\lambda \subset \mathbb{C}[x_1, x_2, x_3, x_4] \), whose vanishing set is an \( S_4 \)-invariant algebraic set inside \( \mathbb{C}^4 \). Describe this vanishing set.

**Exercise 8.8.** Compute all irreducible representations of \( S_4 \) and their dimensions. For \( n \in \{2, 3, 4\} \), write down the decomposition of \( (\mathbb{C}^n)^\otimes 4 \) given by Schur–Weyl duality. Compute the dimensions of \( L_\lambda \) for each summand in this decomposition, and make sure the overall dimensions add up appropriately.

**Exercise 8.9.** Let \( V \) be a rational \( GL_n \) representation. Prove that \( \text{End}_{GL_n}(V) \) is semisimple. Prove that the multiplicity spaces in \( V \) are irreducible modules for \( \text{End}_{GL_n}(V) \). [Hint: What are the irreducible representations of a matrix algebra? Of a product of matrix algebras?]