Lecture 10

Stillman's conjecture via ultraproducts

Stillman's conjecture follows from the existence of small subalgebras (Theorem 9.2) and standard arguments in commutative algebra. In this lecture, we explain these standard arguments and then complete the proof of Stillman's conjecture.

Suppose that R is a ring, x_1 and x_2 are elements of R, and we want to find the projective resolution of $R/(x_1, x_2)$. The first few terms are easy:

$$Re_1 \oplus Re_2 \to R \to R/(x_1, x_2) \to 0$$

where the first map takes e_i to x_i. There is one obvious element in the kernel, namely $x_2e_1 - x_1e_2$. It is thus reasonable to consider the complex

$$\cdots \to 0 \to R \to Re_1 \oplus Re_2 \to R \to R/(x_1, x_2) \to 0.$$

This is not necessarily exact. However, it is the only complex we can really write down without knowing more information about the x's. This "universal" complex is the (augmented) Koszul complex on x_1 and x_2.

We now explain the general construction. Let $x_1, \ldots, x_n$ be elements of R. Let $K_r = \bigwedge^r(R^n)$ and define a differential $d\colon K_r \to K_{r-1}$ by

$$d(e_{\alpha_1} \wedge \cdots \wedge e_{\alpha_r}) = \sum_{i=1}^{r} (-1)^{i+1} x_{\alpha_1} \cdot e_{\alpha_1} \wedge \cdots \wedge \hat{e}_{\alpha_i} \wedge \cdots \wedge e_{\alpha_r}.$$

One readily verifies that $d^2 = 0$ and so $K_\bullet$ is a complex. (Note that $R/(x_1, \ldots, x_n)$ is not a term in this complex.) This is the *Koszul complex* on $x_1, \ldots, x_n$. We write $K(x_1, \ldots, x_n)$ to indicate the dependence on the x's, when needed. The homology of this complex is called *Koszul homology.* Note that $\mathrm{H}_0(K) = R/(x_1, \ldots, x_n)$.

Proposition 10.1. *Let $K' = K(x_1, \ldots, x_{n-1})$ and $K = K(x_1, \ldots, x_n)$. Then we have an exact sequence of complexes*

$$0 \to K' \to K \to K'[-1] \to 0$$

and a long exact sequence in Koszul homology

$$\cdots \longrightarrow \mathrm{H}_i(K') \xrightarrow{x_n} \mathrm{H}_i(K') \longrightarrow \mathrm{H}_i(K) \longrightarrow \mathrm{H}_{i-1}(K') \xrightarrow{x_n} \mathrm{H}_{i-1}(K') \longrightarrow \cdots$$

Proof. We have an obvious inclusion $K' \to K$. The map $K \to K'[-1]$ in degree i is the map $\bigwedge^i(R^n) \to \bigwedge^{i-1}(R^{n-1})$ that takes $e_{\alpha_1} \wedge \cdots \wedge e_{\alpha_{n-1}} \wedge e_n$ to $e_{\alpha_1} \wedge \cdots \wedge e_{\alpha_{n-1}}$, and kills other basis vectors. One easily checks that these are maps of complexes, and it is clear that we have a short exact sequence. The description of the connecting homomorphism in the long exact sequence comes from simply tracing the definitions. $\square$

Corollary 10.2. *Suppose that $x_1, \dots, x_n$ is a regular sequence in R. Then $K = K(x_1, \dots, x_n)$ is exact in positive degrees. In particular, the augmented complex $K \to R/(x_1, \dots, x_n)$ is a projective resolution.*

Proof. Let K' be as above. By induction on n, we see that K' is exact in positive degrees. The long exact sequence above shows that K is exact in degrees ≥ 2. In degree 1, we see that $\mathrm{H}_1(K)$ is the kernel of multiplication by x_n on $\mathrm{H}_0(K') = R/(x_1, \dots, x_{n-1})$. Since we have a regular sequence, this map is injective. $\square$

The following example is extremely useful:

Example 10.3. Let $R = k[x_1, \dots, x_n]$ be a polynomial ring. Then $x_1, \dots, x_n$ is a regular sequence, and $R/(x_1, \dots, x_n) = k$. The Koszul complex gives a projective resolution $\bigwedge^\bullet(R^n) \to k$.

The following is the key result we need, and why regular sequences are so important in our small subalgebras:

Proposition 10.4. *Let R be a graded k-algebra, let $g_1, \dots, g_m$ be a homogeneous regular sequence in R, and let $S = k[t_1, \dots, t_m]$. Consider the k-algebra homomorphism $S \to R$ given by $t_i \mapsto g_i$. Then R is flat over S. In particular, if $\mathfrak{a}$ is an ideal of S then $\mathrm{pdim}_R(R/\mathfrak{a}^e) \leq \mathrm{pdim}_S(S/\mathfrak{a})$.*

Proof. By the above example, we have a projective resolution $K(t_1, \dots, t_m) \to k$ given by the Koszul complex. We thus see that $\mathrm{Tor}^S_\bullet(R, k)$ is computed by the complex $K(t_1, \dots, t_m) \otimes_S R$, which is clearly isomorphic to $K(g_1, \dots, g_m)$. Since $g_1, \dots, g_m$ is a regular sequence, this is exact in positive degrees, and so $\mathrm{Tor}^S_i(R, k) = 0$ for $i > 0$. This implies that R is flat over S.

Now let $\mathfrak{a}$ be an ideal of S. Let $P_\bullet \to S/\mathfrak{a}$ be a projective resolution over S of length $d = \mathrm{pdim}_S(S/\mathfrak{a})$. Then $R \otimes_S P_\bullet \to R \otimes S/\mathfrak{a} = R/\mathfrak{a}^e$ is a projective resolution over R of length d. $\square$

We can now prove Stillman's conjecture.

Theorem 10.5. *Given positive integers $d_1, \dots, d_r$, there exists $N = N(d_1, \dots, d_r)$ such that if $\mathfrak{a}$ is an ideal of $R = k[x_1, \dots, x_n]$ generated by r elements of degrees $d_1, \dots, d_r$ then $\mathrm{pdim}_R(R/\mathfrak{a}) \leq N$.*

Proof. Let $f_1, \dots, f_r$ in $k[x_1, \dots, x_n]$ of degrees $d_1, \dots, d_r$ be given. By existence of small subalgebras (Theorem 9.2). the f_i's belong to some $S = k[g_1, \dots, g_s]$ where the g_i's are a regular sequence and $s \leq C = C(d_1, \dots, d_r)$; note that $S \cong k[t_1, \dots, t_s]$. Let $\mathfrak{a}_0$ be the ideal of $k[g_1, \dots, g_s]$ generated by the f_i's, so that $\mathfrak{a}$ is the extension of $\mathfrak{a}_0$. By the previous proposition, $\mathrm{pdim}_R(R/\mathfrak{a}) \leq \mathrm{pdim}_S(S/\mathfrak{a}_0)$, and the latter is at most s by Hilbert's syzygy theorem. We can thus take $N = C$. $\square$

Exercises

Exercise 10.1. We have seen that if $f_1, \dots, f_r$ is a regular sequence then $f_1, \dots, f_r$ is algebraically independent. Given an example of $f_1, \dots, f_r \in k[x_1, \dots, x_n]$ that are homogeneous of positive degree and algebraically independent, but not a regular sequence.

Exercise 10.2. Let R be a graded ring (in non-negative degrees) with $R_0 = k$ a field. Let $f_1, \dots, f_r$ be homogeneous elements of positive degree such that we have $\mathrm{H}_1(K(f_1, \dots, f_r)) = 0$. Show that $f_1, \dots, f_r$ is a regular sequence.

Exercise 10.3. Describe the top Koszul homology group $\mathrm{H}_n(K(x_1, \dots, x_n))$, for arbitrary elements $x_1, \dots, x_n$ in a ring R.