LECTURE 11

Representations of $\mathbf{GL}_\infty$

We work over the complex numbers in the remaining lectures. Let F be a polynomial functor. We can "probe" F by evaluating it on $\mathbf{C}^n$. This might give us zero, or something non-zero but still degenerate, if n is too small. If F is a finite sum of Schur functors, the above theorem essentially tells us that if n is sufficiently large, we can see "enough" by evaluating on $\mathbf{C}^n$. However, if F is an infinite sum of Schur functors, we might never see the full picture by evaluating on $\mathbf{C}^n$.

EXAMPLE 11.1. Let $F = \bigoplus_{m \geq 0} \bigwedge^m$. This is an infinite sum of Schur functors. However,

$$F(\mathbf{C}^n) = \bigoplus_{m=0}^{n} \bigwedge^m(\mathbf{C}^n)$$

is a finite sum of irreducible representations of $\mathbf{GL}_n$. Thus we never see all of F by evaluating on $\mathbf{C}^n$.

EXAMPLE 11.2. Recall from Lecture 8 that $R(V) = \mathrm{Sym}(\mathrm{Sym}^2(V))$ is the coordinate ring of the space of symmetric bilinear forms on V, and $\mathfrak{a}_r(V) \subset R(V)$ is the ideal defining the rank $\leq r$ locus. Every symmetric bilinear form on $\mathbf{C}^n$ has rank at most n, and so $\mathfrak{a}_r(\mathbf{C}^n) = 0$ for $n \geq r$. This shows that we cannot distinguish the various $\mathfrak{a}_r$'s (as subfunctors of R) by evaluating on a single $\mathbf{C}^n$.

This above discussion suggests it might be helpful to evaluate polynomial functors on infinite dimensional spaces. We now explain how this works.

Let $\mathbf{V} = \bigcup_{n \geq 1} \mathbf{C}^n$ and let $\mathbf{GL} = \bigcup_{n \geq 1} \mathbf{GL}_n$. Note that $\mathbf{V}$ is naturally a representation of $\mathbf{GL}$; we call it the *standard representation*. Using the fact that $\mathbf{S}_{\underline{\lambda}}(\mathbf{C}^n)$ is an irreducible of $\mathbf{GL}_n$ for large n, it is not hard to see that $\mathbf{S}_{\underline{\lambda}}(\mathbf{V})$ is an irreducible representation of $\mathbf{GL}$. We say that a representation of $\mathbf{GL}$ is *polynomial* if it is isomorphic to a (perhaps infinite) direct sum of representations of the form $\mathbf{S}_{\underline{\lambda}}(\mathbf{V})$. The category $\mathrm{Rep}^{\mathrm{pol}}(\mathbf{GL})$ of polynomial representations is a semi-simple abelian category, with the simple objects being the $\mathbf{S}_{\underline{\lambda}}(\mathbf{V})$, and is closed under tensor product.

It is not difficult to deduce from the above discussion that the functor

$$\begin{aligned} \{\text{polynomial functors}\} &\to \mathrm{Rep}^{\mathrm{pol}}(\mathbf{GL}) \\ F &\mapsto F(\mathbf{V}) \end{aligned}$$

is an equivalence of categories. In other words, evaluating a polynomial functor on $\mathbf{V}$ does not lose information (assuming we keep track of the resulting $\mathbf{GL}$ action).

The previous paragraph shows that one can pass back and forth from polynomial functors to polynomial representations. Polynomial representations have some advantages over polynomial functors. For one, they are more concrete: there is just one vector space and one group action to keep track of. Another advantage is that one can apply ideas from the representation theory of $\mathbf{GL}_n$ to polynomial representations. In particular, one has a theory of weights, which we now explain.

Let $\mathbf{T}$ be the subgroup of $\mathbf{GL}$ consisting of diagonal matrices. A *weight* of $\mathbf{T}$ is a tuple $\lambda = (\lambda_1, \lambda_2, \ldots)$ of integer such that $\lambda_i = 0$ for $i \gg 0$. Given λ, we let $\chi_\lambda \colon \mathbf{T} \to \mathbf{C}^\times$ be the homomorphism defined by

$$\chi_\lambda(t) = t_1^{\lambda_1} t_2^{\lambda_2} \cdots .$$

Here t is a diagonal matrix and t_1, t_2, and so on are its diagonal entries. Suppose that V is a polynomial representation. We define the λ-weight space V_λ to be the set of vectors $v \in V$ such that $t \cdot v = \chi_\lambda(t) \cdot v$ for all $t \in \mathbf{T}$. One can show that V is the direct sum of its weight spaces V_λ as λ varies.

Exercises

Exercise 11.1. Let V be a finite length polynomial representation of $\mathbf{GL}$. Show that the infinite symmetric group $\mathfrak{S}$ acts with finitely many orbits on the set of weights appearing in V.

Exercise 11.2. Let V be a polynomial representation of $\mathbf{GL}$ and let V_{1^k} denote its 1^k weight space under $\mathbf{T}$. (1^k means the weight $(1, \ldots, 1, 0, 0, \ldots)$, where there are k 1's.)

(a) Show that V_{1^k} is naturally a representation of the symmetric group $\mathfrak{S}_k$.
(b) When $V = \mathbf{S}_\lambda(\mathbf{V})$ show that V_{1^k} is the Specht module $\mathbf{Sp}_\lambda$.

Additional exercises

Exercise 11.3. Let $\mathrm{Rep}(\mathfrak{S}_*)$ be the category whose objects are sequences $(M_n)_{n \geq 0}$ where M_n is a complex representation of the symmetric group $\mathfrak{S}_n$.

(a) Show that there is a unique (up to isomorphism) equivalence of categories

$$\Phi \colon \mathrm{Rep}(\mathfrak{S}_*) \to \mathrm{Rep}^{\mathrm{pol}}(\mathbf{GL})$$

satisfying $\Phi(\mathbf{Sp}_\lambda) = \mathbf{S}_\lambda(\mathbf{V})$. (Here we regard $\mathbf{Sp}_\lambda$ as the sequence that is only non-zero in index $|\lambda|$.)
(b) Via Φ, we can transport the tensor product on $\mathrm{Rep}^{\mathrm{pol}}(\mathbf{GL})$ to $\mathrm{Rep}(\mathfrak{S}_*)$. What do we get?

Notes

See [SS] for more background on polynomial representations.