Lecture 12

GL-varieties

A $\mathbf{GL}$-*algebra* is a $\mathbf{C}$-algebra equipped with an action of $\mathbf{GL}$ by algebra automorphisms under which it forms a polynomial representation. A simple example is $\mathrm{Sym}(V)$, where V is a polynomial representation of $\mathbf{GL}$. We say that a $\mathbf{GL}$-algebra R is *finitely* $\mathbf{GL}$-*generated* if it is generated as a k-algebra by the $\mathbf{GL}$-orbits of finitely many elements; equivalently, R is a quotient of $\mathrm{Sym}(V)$ for some finite length polynomial representation V. We now come to an important definition:

Definition 12.1. An *affine* $\mathbf{GL}$-*variety* is a scheme of the form $\mathrm{Spec}(R)$ where R is a reduced and finitely $\mathbf{GL}$-generated $\mathbf{GL}$-algebra.

Note that $\mathbf{GL}$-varieties are almost always infinite dimensional. The following definition introduces the most important $\mathbf{GL}$-varieties:

Definition 12.2. For a partition λ, define $\mathbf{A}^{\lambda}$ to be the spectrum of the $\mathbf{GL}$-algebra $\mathrm{Sym}(\mathbf{S}_{\lambda})$. For a tuple $\underline{\lambda} = [\lambda_1, \ldots, \lambda_n]$ of partitions, put $\mathbf{A}^{\underline{\lambda}} = \mathbf{A}^{\lambda_1} \times \cdots \times \mathbf{A}^{\lambda_n}$.

These $\mathbf{GL}$-varieties play the role of the familiar $\mathbf{A}^n$'s in classical algebraic geometry: every affine $\mathbf{GL}$-variety is a closed $\mathbf{GL}$-subvariety of some $\mathbf{A}^{\underline{\lambda}}$. Let's look at some special cases now.

Example 12.3. Consider the case $\lambda = (d)$, so that $\mathbf{S}_{\lambda} = \mathrm{Sym}^d$. Then $\mathbf{A}^{\lambda}$ is the spectrum of the ring $\mathrm{Sym}(\mathrm{Sym}^d)$, and thus identified with the dual space $(\mathrm{Sym}^d)^*$. This, in turn, is naturally identified with the degree d piece of the inverse limit ring $\mathbf{R}$. To be completely concrete, a point in $\mathbf{A}^{(d)}$ can be written in the form $\sum_{|\alpha|=d} c_{\alpha} x^{\alpha}$, where the sum is over exponent vectors α and $c_{\alpha} \in k$, and a closed subset is defined by polynomial equations in the coefficients (the c's).

Example 12.4. Let $\underline{\lambda} = [(d_1), \ldots, (d_r)]$. Then $\mathbf{A}^{\underline{\lambda}}$ parametrizes tuples $(f_1, \ldots, f_r) \in \mathbf{R}$ where f_i is homogeneous of degree d_i. Thus $\mathbf{A}^{\underline{\lambda}}$ is a kind of moduli space for (generators of) ideals in the Stillman regime. At each point in $\mathbf{A}^{\underline{\lambda}}$, one can consider the projective dimension of the corresponding ideal, and Stillman's conjecture roughly asserts that this function is bounded. Thus one might hope to prove Stillman's conjecture (and similar statements) by understanding aspects of $\mathbf{GL}$-varieties. We pursue this idea in Lectures 15 and 16.

If you're puzzled by what a general $\mathbf{A}^{\lambda}$ looks like, you can simply focus on $\mathbf{A}^{(d)}$'s without losing too much. We now give some examples of more interesting $\mathbf{GL}$-varieties.

Example 12.5. We have seen (Exercise 3.4) that for degree 2 elements in $\mathbf{R}$, the condition "strength $\leq s$" is described by polynomial equations on the coefficients.

Thus the strength $\leq s$ locus in $\mathbf{A}^{(2)}$ is a closed $\mathbf{GL}$-subvariety for any fixed s. (For elements of $\mathbf{A}^{(2)}$, rank and strength are closely related. If we had instead used the rank $\leq r$ locus, the defining ideal would be $\mathfrak{a}_r(\mathbf{V})$, where $\mathfrak{a}_r$ is as in Example 11.2.)

EXAMPLE 12.6. One can consider the strength $\leq s$ locus in $\mathbf{A}^{(d)}$. It is known that for $d \geq 4$ this locus is not Zariski closed [BBOV]. Its Zariski closure is a closed $\mathbf{GL}$-subvariety of $\mathbf{A}^{(d)}$. The theory of $\mathbf{GL}$-varieties developed in [BDES] gives tools for understanding examples like this one.

Exercises

Exercise 12.1. Describe all closed $\mathbf{GL}$-subvarieties of $\mathbf{A}^{(1)} \times \mathbf{A}^{(1)}$.

Exercise 12.2. Show that the rank $\leq r$ loci account for all the non-empty proper closed $\mathbf{GL}$-subvarieties of $\mathbf{A}^{(2)}$.

EXAMPLE 12.7. Let $\underline{\lambda} = [(d_1), \ldots, (d_r)]$. Show that any map of $\mathbf{GL}$-varieties $\varphi \colon \mathbf{A}^{\underline{\lambda}} \to \mathbf{A}^{(e)}$ has the form

$$\varphi(f_1, \ldots, f_r) = \Phi(f_1, \ldots, f_r)$$

where $\Phi \in k[T_1, \ldots, T_r]$ is polynomial. Moreover, show that Φ is homogeneous of degree e if T_i is given degree d_i. (A map of $\mathbf{GL}$-varieties is simply a $\mathbf{GL}$-equivariant map of schemes over k.)

Additional exercises

Exercise 12.3. A point in a $\mathbf{GL}$-variety is $\mathbf{GL}$-*generic* if it has dense $\mathbf{GL}$-orbit.

(a) Show that any element of $\mathbf{A}^{(2)}$ of infinite strength is $\mathbf{GL}$-generic. In fact, this is true in $\mathbf{A}^{(d)}$ as well, but the proof is much harder.
(b) Let $f = \sum_{i \geq 0} x_{3i+1}x_{3i+2}x_{3i+3}$, regarded as a point of $\mathbf{A}^{(3)}$. Show that f is $\mathbf{GL}$-generic. (Hint: show that any cubic in n variables can be obtained as a limit of points in the $\mathbf{GL}$-orbit of f.)
(c) On the other hand, show that there is no point in $\mathrm{Sym}^3(k^n)$ with dense $\mathbf{GL}_n$-orbit. Thus the above result is somewhat surprising.
(d) Show that if $\underline{\lambda}$ is any tuple of non-empty partitions the space $\mathbf{A}^{\underline{\lambda}}$ admits a $\mathbf{GL}$-generic point.

Exercise 12.4. Let $X = \mathrm{Spec}(R)$ be an irreducible $\mathbf{GL}$-variety. The *invariant function field* of X is the field $k(X)^{\mathbf{GL}}$ of $\mathbf{GL}$-invariant elements in the function field $k(X) = \mathrm{Frac}(R)$. Compute this when X is the closed subvariety of $\mathbf{A}^{(1)} \times \mathbf{A}^{(1)}$ consisting of pairs (x, y) that are linearly dependent.

Notes

The notion of $\mathbf{GL}$-variety was formally introduced in [BDES, §2], though the idea had been in the air for a little while (e.g., [Dr] is about $\mathbf{GL}$-varieties but doesn't use that term).