Lecture 13

The embedding and shift theorems

A common theme in representation stability is that objects can be made simpler by applying a "shift" operation. This is true of **GL**-varieties, as we will see in this lecture. These results give us an inductive way to understand **GL**-varieties, and are among the most powerful tools we have.

We first define the shift operation. Let $G(n)$ be the subgroup of $\mathbf{GL}_n$ consisting of matrices of the form

$$\begin{pmatrix} 1 & 0 \\ 0 & * \end{pmatrix},$$

where the top left block is $n \times n$. Of course, $G(n)$ is itself isomorphic to **GL**. Given an object X on which **GL** acts, we can thus restrict to $G(n)$ and then identify $G(n)$ with **GL** to obtain a new action of **GL**. This is called the nth *shift* of X, and denoted $\mathrm{Sh}_n(X)$. For example, we have $\mathrm{Sh}_n(\mathbf{V}) = \mathbf{C}^n \oplus \mathbf{V}$, and one can use this to figure out shifts of Schur functors, e.g.,

$$\mathrm{Sh}_n(\mathrm{Sym}^2(\mathbf{V})) = \mathrm{Sym}^2(\mathbf{C}^n \oplus \mathbf{V}) = \mathrm{Sym}^2(\mathbf{C}^n) \oplus (\mathbf{C}^n \otimes \mathbf{V}) \oplus \mathrm{Sym}^2(\mathbf{V}).$$

The following is the *embedding theorem*:

Theorem 13.1. *Let Y be a* **GL***-variety, let λ be a non-empty partition, and let X be a closed* **GL***-subvariety of $Y \times \mathbf{A}^{\lambda}$. Then one of the following two possibilities holds:*

(a) $X = Y_0 \times \mathbf{A}^{\lambda}$ for some closed **GL***-subvariety $Y_0 \subset Y$; or*
(b) there is a non-empty open subset of $\mathrm{Sh}_n(X)$, for some n, that embeds into $\mathrm{Sh}_n(Y) \times \mathbf{A}^{\underline{\mu}}$ for some $\underline{\mu}$, where every partition in $\underline{\mu}$ is smaller than λ.

Before giving the proof, we illustrate the main idea in a special case.

Example 13.2. Let Y be a point, let $\lambda = (2)$, and let X be the rank ≤ 1 locus in $\mathbf{A}^{(2)}$. We think of elements of $\mathbf{A}^{(2)}$ as infinite symmetric matrices x, and write $x_{i,j}$ for the (i,j) entry. Let $U \subset X$ be the open set where $x_{1,1}$ is non-zero; this is $G(1)$-stable. Suppose $x \in U$. Then x has rank 1, and the first column is a basis for its column space. For $i > 1$, the ith column of x is a scalar multiple of the first column, and by looking at the first row we see that the scalar is $x_{i,1}/x_{1,1}$. Thus we can solve for all entries in x in terms of the first row. This shows that

$$\mathrm{Sh}_1(U) \cong \mathbf{C}^{\times} \times \mathbf{A}^{(1)}$$

as **GL**-varieties. Here the $\mathbf{C}^{\times}$ is the $x_{1,1}$ coordinate, and the $\mathbf{A}^{(1)}$ records the first row with $x_{1,1}$ omitted. (Note that the first row with $x_{1,1}$ omitted looks like $\mathbf{A}^{(1)}$ but with $G(1)$-acting; after shifting, we actually get $\mathbf{A}^{(1)}$ with **GL** acting.)

Proof of Theorem 13.1. The main idea is like the example: we will construct a function h such that on the $h \neq 0$ locus we can solve for many of the coordinates in terms of simpler coordinates. This will produce an embedding of the kind we want. We argue as follows:

- We prove the theorem just for $\lambda = (2)$ for simplicity; the general argument is exactly the same, just with more complicated notation.
- Let R be the coordinate ring of Y, so that $R[x_{i,j}]$ is the coordinate ring of $Y \times \mathbf{A}^{\lambda}$ (where $x_{i,j} = x_{j,i}$). Let $I \subset R[x_{i,j}]$ be the ideal for X, let J_0 be its contraction to R, and let J be the extension of J. We have $J \subset I$ with equality if and only if we're in case (a). Assume we're not in case (a), so I is strictly larger than J.
- We have seen that polynomial representation of $\mathbf{GL}_\infty$ are determined by their 1^n weight spaces. Thus the 1^n weight space of I is strictly larger than that for J, for some n; let f be such a weight vector in I that's not in J.
- A 1^n weight vector in $R[x_{i,j}]$ can be written as a sum of terms of the form $x_{i_1,j_1} \cdots x_{i_r,j_r} g$ where all indices are distinct and g is an 1^S-weight vector of R, where $S = [n] \setminus \{i_1, j_1, \ldots, i_r, j_r\}$. Applying a permutation to f, we can thus assume that $f = hx_{n-1,n} + g$, where h is a non-zero 1^{n-2}-weight vector in $R[x_{i,j}]$ and the variable $x_{n-1,n}$ does not appear in g.
- In $(R[x_{i,j}]/I)[1/h]$, we have $x_{n-1,n} = -g/h$. The variables appearing in the right side are of the form $x_{i,j}$ or $x_{n,i}$ or $x_{n-1,i}$ where $i, j \leq n-2$; call these "small." Thus applying permutations of $\{n-1, n, n+1, \ldots\}$, we see that every $x_{i,j}$ can be expressed in terms of small variables in this ring. In other words, we have a $G(n)$-equivariant surjection

$$(R[x_{i,j}]_{1 \leq i,j \leq n-2}[1/h]) \otimes k[y_i, z_i]_{i \geq n-1} \to (R[x_{i,j}]/I)[1/h]$$

  where y_i maps to $x_{n-1,i}$ and z_i to $x_{n,i}$. Thus case (b) holds with $\underline{\mu} = [(1), (1)]$. □

The *shift theorem* is the following. For a function h on a variety X, we let $X[1/h]$ be the non-vanishing locus of h.

THEOREM 13.3. *Let X be a $\mathbf{GL}$-variety. Then there is $n \geq 0$ and a non-zero $\mathbf{GL}$-invariant function h on $\mathrm{Sh}_n(X)$ such that $\mathrm{Sh}_n(X)[1/h]$ is isomorphic, as a $\mathbf{GL}$-variety, to $B \times \mathbf{A}^{\underline{\rho}}$, where B is an ordinary (finite dimensional) variety and $\underline{\rho}$ is a tuple of partitions.*

Proof. Embed X into $\mathbf{A}^{\underline{\mu}}$ for some $\underline{\mu}$. We proceed by induction on $\underline{\mu}$. If $\underline{\mu}$ only consists of empty partitions, the result is clear (we don't need to shift or pass to an open set: we can just take $B = X$ and $\underline{\rho}$ to be empty). This is the base case of the induction.

Suppose now that $\underline{\mu}$ contains some non-empty partition. Let N be the maximal size of a partition in $\underline{\mu}$, let λ be a partition in $\underline{\mu}$ of size N, let $\underline{\nu}$ be the reamining part of $\underline{\mu}$, and let $Y = \mathbf{A}^{\underline{\nu}}$. We have $X \subset Y \times \mathbf{A}^{\lambda}$, so we are in the setting of the embedding theorem. In case (i), we have $X = Y_0 \times \mathbf{A}^{\lambda}$ for some $Y_0 \subset \mathbf{A}^{\underline{\nu}}$. Since $\underline{\nu}$ is smaller than $\underline{\mu}$, the shift theorem holds for Y_0 by induction; it is easy to see that it then holds for X. Now suppose we're in case (ii). Then after shifting and passing to an open set, X embeds into $\mathrm{Sh}_n(Y) \times \mathbf{A}^{\underline{\sigma}}$, where every partition in $\underline{\sigma}$ is smaller than λ. This space has the form $\mathbf{A}^{\underline{\tau}}$, where $\underline{\tau}$ is smaler than $\underline{\mu}$. (All partitions in $\underline{\tau}$ have size at most N, and the number of partitions in $\underline{\tau}$ of size N is one less than the number in $\underline{\mu}$ of size N.) Thus by induction, the shift theorem holds for subvarieties of $\mathbf{A}^{\underline{\tau}}$, and so the result follows. $\square$

Exercises

Exercise 13.1. Let X be the closed $\mathbf{GL}$-subvariety of $\mathbf{A}^{[(1),(1)]}$ consisting of linearly dependent pairs. Explicitly verify the conclusion of the shift theorem in this case.

Exercise 13.2. Let X be the rank $\le r$ locus in $\mathbf{A}^{(2)}$. Explicitly verify the conclusion of the shift theorem in this case.

Exercise 13.3. Show that $\mathrm{Sh}_n(\mathbf{S}_\lambda)$ has the form $\mathbf{S}_\lambda \oplus \cdots$, where the remaining terms are Schur functors of smaller degree.

Additional exercises

Exercise 13.4. Let X be a $\mathbf{GL}$-variety. Show that the invariant function field $k(X)^{\mathbf{GL}}$ is a finitely generated extension of k. (See Exercise 12.4 for the definition of $k(X)^{\mathbf{GL}}$.) [Hint: use the shift theorem (Theorem 13.3).]

Exercise 13.5. Explicitly compute $\mathrm{Sh}_n(\mathbf{S}_\lambda)$ in terms of Littlewood–Richardson coefficients (if you know what these are).

Exercise 13.6. Let X be an affine $\mathbf{GL}$-variety.

(a) Show that there is a natural surjective map of $\mathbf{GL}$-varieties $\mathrm{Sh}_n(X) \to X$. [Hint: this is induced by the canonical inclusion $\mathbf{V} \to \mathrm{Sh}_n(\mathbf{V})$.]

(b) Show that there is a dominant morphism $B \times \mathbf{A}^{\underline{\lambda}} \to X$ for some finite dimensional variety B and some tuple $\underline{\lambda}$. [This says that X is "unirational up to a finite dimensional error."]

Notes

The embedding theorem appeared implicity in [Dr]. It was isolated as a standalone result in [BDES, §4] when it was realized how useful it can be. The shift theorem was proved in [BDES, §5]. See [BDES] for more details on the proofs.