A common theme in representation stability is that objects can be made simpler by applying a “shift” operation. This is true of \( \text{GL} \)-varieties, as we will see in this lecture. These results give us an inductive way to understand \( \text{GL} \)-varieties, and are among the most powerful tools we have.

We first define the shift operation. Let \( G(n) \) be the subgroup of \( \text{GL}_n \) consisting of matrices of the form 
\[
\begin{pmatrix}
1 & 0 \\
0 & *
\end{pmatrix},
\]
where the top left block is \( n \times n \). Of course, \( G(n) \) is itself isomorphic to \( \text{GL}_n \). Given an object \( X \) on which \( \text{GL} \) acts, we can thus restrict to \( G(n) \) and then identify \( G(n) \) with \( \text{GL}_n \) to obtain a new action of \( \text{GL} \). This is called the \( n \)th shift of \( X \), and denoted \( \text{Sh}_n(X) \). For example, we have \( \text{Sh}_n(V) = C^n \oplus V \), and one can use this to figure out shifts of Schur functors, e.g.,
\[
\text{Sh}_n(\text{Sym}^2(V)) = \text{Sym}^2(C^n \oplus V) = \text{Sym}^2(C^n) \oplus (C^n \otimes V) \oplus \text{Sym}^2(V).
\]

The following is the embedding theorem:

**Theorem 13.1.** Let \( Y \) be a \( \text{GL} \)-variety, let \( \lambda \) be a non-empty partition, and let \( X \) be a closed \( \text{GL} \)-subvariety of \( Y \times A^\lambda \). Then one of the following two possibilities holds:

(a) \( X = Y_0 \times A^\lambda \) for some closed \( \text{GL} \)-subvariety \( Y_0 \subset Y \); or
(b) there is a non-empty open subset of \( \text{Sh}_n(X) \), for some \( n \), that embeds into \( \text{Sh}_n(Y) \times A^\mu \) for some \( \mu \), where every partition in \( \mu \) is smaller than \( \lambda \).

Before giving the proof, we illustrate the main idea in a special case.

**Example 13.2.** Let \( Y \) be a point, let \( \lambda = (2) \), and let \( X \) be the rank \( \leq 1 \) locus in \( A^{(2)} \). We think of elements of \( A^{(2)} \) as infinite symmetric matrices \( x \), and write \( x_{i,j} \) for the \((i,j)\) entry. Let \( U \subset X \) be the open set where \( x_{1,1} \) is non-zero; this is \( G(1) \)-stable. Suppose \( x \in U \). Then \( x \) has rank 1, and the first column is a basis for its column space. For \( i > 1 \), the \( i \)th column of \( x \) is a scalar multiple of the first column, and by looking at the first row we see that the scalar is \( x_{1,i}/x_{1,1} \). Thus we can solve for all entries in \( x \) in terms of the first row. This shows that \( \text{Sh}_1(U) \cong C^x \times A^{(1)} \)

as \( \text{GL} \)-varieties. Here the \( C^x \) is the \( x_{1,1} \) coordinate, and the \( A^{(1)} \) records the first row with \( x_{1,1} \) omitted. (Note that the first row with \( x_{1,1} \) omitted looks like \( A^{(1)} \) but with \( G(1) \)-acting; after shifting, we actually get \( A^{(1)} \) with \( \text{GL} \) acting.)
Proof of Theorem 13.1. The main idea is like the example: we will construct a function \( h \) such that on the \( h \neq 0 \) locus we can solve for many of the coordinates in terms of simpler coordinates. This will produce an embedding of the kind we want. We argue as follows:

- We prove the theorem just for \( \lambda = (2) \) for simplicity; the general argument is exactly the same, just with more complicated notation.
- Let \( R \) be the coordinate ring of \( Y \), so that \( R[x_{i,j}] \) is the coordinate ring of \( Y \times A^\lambda \) (where \( x_{i,j} = x_{j,i} \)). Let \( I \subset R[x_{i,j}] \) be the ideal for \( X \), let \( J_0 \) be its contraction to \( R \), and let \( J \) be the extension of \( J_0 \). We have \( J \subset I \) with equality if and only if we’re in case (a). Assume we’re not in case (a), so \( I \) is strictly larger than \( J \).
- We have seen that polynomial representation of \( GL_\infty \) are determined by their \( 1^n \) weight spaces. Thus the \( 1^n \) weight space of \( I \) is strictly larger than that for \( J_0 \) for some \( n \); let \( f \) be such a weight vector in \( I \) that’s not in \( J \).
- A \( 1^n \) weight vector in \( R[x_{i,j}] \) can be written as a sum of terms of the form \( x_{i_1,j_1} \cdots x_{i_r,j_r}g \) where all indices are distinct and \( g \) is an \( 1^S \)-weight vector of \( R \), where \( S = [n] \setminus \{i_1,j_1, \ldots, i_r,j_r\} \). Applying a permutation to \( f \), we can thus assume that \( f = hx_{n-1,n} + g \), where \( h \) is a non-zero \( 1^{n-2} \)-weight vector in \( R[x_{i,j}] \) and the variable \( x_{n-1,n} \) does not appear in \( g \).
- In \( (R[x_{i,j}]/I)[1/h] \), we have \( x_{n-1,n} = -g/h \). The variables appearing in the right side are of the form \( x_{i,j} \) or \( x_{n,i} \) or \( x_{n-1,i} \) where \( i, j \leq n-2 \); call these “small.” Thus applying permutations of \( \{n-1, n, n+1, \ldots\} \), we see that every \( x_{i,j} \) can be expressed in terms of small variables in this ring. In other words, we have a \( G(n) \)-equivariant surjection

\[
(R[x_{i,j}]_{1 \leq i,j \leq n-2}[1/h]) \otimes k[y_i,z_i]_{i \geq n-1} \rightarrow (R[x_{i,j}]/I)[1/h]
\]

where \( y_i \) maps to \( x_{n-1,i} \) and \( z_i \) to \( x_{n,i} \). Thus case (b) holds with \( \underline{\mu} = [(1),(1)] \).

The shift theorem is the following. For a function \( h \) on a variety \( X \), we let \( X[1/h] \) be the non-vanishing locus of \( h \).

Theorem 13.3. Let \( X \) be a \( GL \)-variety. Then there is \( n \geq 0 \) and a non-zero \( GL \)-invariant function \( h \) on \( Sh_n(X) \) such that \( Sh_n(X)[1/h] \) is isomorphic, as a \( GL \)-variety, to \( B \times A^\underline{\mu} \), where \( B \) is an ordinary (finite dimensional) variety and \( \underline{\mu} \) is a tuple of partitions.

Proof. Embed \( X \) into \( A^\underline{\mu} \) for some \( \underline{\mu} \). We proceed by induction on \( \underline{\mu} \). If \( \underline{\mu} \) only consists of empty partitions, the result is clear (we don’t need to shift or pass to an open set: we can just take \( B = X \) and \( \underline{\rho} \) to be empty). This is the base case of the induction.

Suppose now that \( \underline{\mu} \) contains some non-empty partition. Let \( N \) be the maximal size of a partition in \( \underline{\mu} \), let \( \lambda \) be a partition in \( \underline{\mu} \) of size \( N \), let \( \underline{\nu} \) be the remaining
part of $\mu$ and let $Y = A^\mu$. We have $X \subset Y \times A^\lambda$, so we are in the setting of the embedding theorem. In case (i), we have $X = Y_0 \times A^\lambda$ for some $Y_0 \subset A^\mu$. Since $\nu$ is smaller than $\mu$, the shift theorem holds for $Y_0$ by induction; it is easy to see that it then holds for $X$. Now suppose we’re in case (ii). Then after shifting and passing to an open set, $X$ embeds into $\text{Sh}_n(Y) \times A^\mu$, where every partition in $\sigma$ is smaller than $\lambda$. This space has the form $A^\tau$, where $\tau$ is smaller than $\mu$. (All partitions in $\tau$ have size at most $N$, and the number of partitions in $\tau$ of size $N$ is one less than the number in $\mu$ of size $N$.) Thus by induction, the shift theorem holds for subvarieties of $A^\tau$, and so the result follows. 

**Exercises**

**Exercise 13.1.** Let $X$ be the closed $\text{GL}$-subvariety of $A^{[(1),(1)]}$ consisting of linearly dependent pairs. Explicitly verify the conclusion of the shift theorem in this case.

**Exercise 13.2.** Let $X$ be the rank $\leq r$ locus in $A^{(2)}$. Explicitly verify the conclusion of the shift theorem in this case.

**Exercise 13.3.** Show that $\text{Sh}_n(S_\lambda)$ has the form $S_\lambda \oplus \cdots$, where the remaining terms are Schur functors of smaller degree.

**Additional exercises**

**Exercise 13.4.** Let $X$ be a $\text{GL}$-variety. Show that the invariant function field $k(X)^{\text{GL}}$ is a finitely generated extension of $k$. (See Exercise 12.4 for the definition of $k(X)^{\text{GL}}$.) [Hint: use the shift theorem (Theorem 13.3).]

**Exercise 13.5.** Explicitly compute $\text{Sh}_n(S_\lambda)$ in terms of Littlewood–Richardson coefficients (if you know what these are).

**Exercise 13.6.** Let $X$ be an affine $\text{GL}$-variety.

(a) Show that there is a natural surjective map of $\text{GL}$-varieties $\text{Sh}_n(X) \to X$. [Hint: this is induced by the canonical inclusion $V \to \text{Sh}_n(V)$.]  

(b) Show that there is a dominant morphism $B \times A^\lambda \to X$ for some finite dimensional variety $B$ and some tuple $\lambda$. [This says that $X$ is “unirational up to a finite dimensional error.”]

**Notes**

The embedding theorem appeared implicitly in [Dr]. It was isolated as a standalone result in [BDES, §4] when it was realized how useful it can be. The shift theorem was proved in [BDES, §5]. See [BDES] for more details on the proofs.