Lecture 14

Draisma's theorem

Recall that a topological space is *noetherian* if it satisfies DCC on closed subsets. If X is an algebraic variety then its underlying topological space (with the Zariski topology) is noetherian; this is a weak version of the Hilbert basis theorem.

If X is a $\mathbf{GL}$-variety then its underlying topological space is essentially never noetherian, for if we ignore the $\mathbf{GL}$ action then we can get things like infinite affine space. So if we want some finiteness, we need to take into account the group action. This is what the following definition does:

Definition 14.1. Suppose a group G acts on a topological space X. We say that X is *(topologically) G-noetherian*[3] if it satisfies DCC on G-stable closed subsets.

Example 14.2. The $\mathbf{GL}$-variety $\mathbf{A}^{(2)}$ is $\mathbf{GL}$-noetherian. Indeed, for $s \geq 0$ let $Z_s \subset \mathbf{A}^{(2)}$ be the rank $\leq s$ locus. We have seen (Exercise 12.2) that the Z_s are closed and that they account for all non-empty proper closed $\mathbf{GL}$-subvarieties. We thus see that the lattice of closed $\mathbf{GL}$-subvarieties of $\mathbf{A}^{(2)}$ is a single chain:

$$\varnothing \subset Z_0 \subset Z_1 \subset \cdots \subset Z_s \subset \cdots \subset \mathbf{A}^{(2)}.$$

This clearly satisfies DCC.

In 2017, Draisma proved the following important theorem, which vastly generalizes the above example:

Theorem 14.3. *An affine $\mathbf{GL}$-variety is $\mathbf{GL}$-noetherian.*

Proof. Draisma's proof is a rather involved induction argument. We present the main idea in a special case to circumvent much of inductive baggage. Specifically, we show that $\mathbf{A}^{\lambda}$ is $\mathbf{GL}$-noetherian for a non-empty partition λ, assuming that $\mathbf{A}^{\underline{\mu}}$ is $\mathbf{GL}$-noetherian whenever every partition in $\underline{\mu}$ is smaller than λ. Here is the argument:

- Given a proper closed $\mathbf{GL}$-subvariety X of $\mathbf{A}^{\lambda}$, let δ_X be the minimal degree of an element of its ideal. We'll show that every proper closed $\mathbf{GL}$-subvariety X of $\mathbf{A}^{\lambda}$ is $\mathbf{GL}$-noetherian by induction on δ_X, which clearly implies that $\mathbf{A}^{\lambda}$ is $\mathbf{GL}$-noetherian. The base case $\delta_X = 0$ is trivial (for then $X = \varnothing$).
- Thus suppose $\delta_X > 0$ and let f be an element of the ideal of X of degree δ_X. By the embedding theorem, there is a non-zero invariant function h on $\mathrm{Sh}_n(X)$, for some n, such that $\mathrm{Sh}_n(X)[1/h]$ embeds into $\mathbf{A}^{\underline{\mu}}$ for some tuple $\underline{\mu}$ in which all parts are smaller than λ. By the *proof* of the embedding theorem, we can take h to have degree strictly smaller than that of f.

[3] In many cases, X will be a scheme, and then there might be stronger versions of G-noetherianity one is interested in; this is why the word "topologically" is sometimes used.

- By assumption, $\mathbf{A}^{\underline{\mu}}$ is $\mathbf{GL}$-noetherian. It follows that the closed subvariety $\mathrm{Sh}_n(X)[1/h]$ is $\mathbf{GL}$-noetherian by Exercise 14.1(a). This means that $X[1/h]$ is $G(n)$-noetherian. By Exercise 14.1(c), we see that $U = \bigcup_{g \in \mathbf{GL}} gX[1/h]$ is $\mathbf{GL}$-noetherian. (Note that U is a quasi-affine $\mathbf{GL}$-variety.)
- Let $Z \subset X$ by the common zero locus of the $\mathbf{GL}$-orbit of h. This is a closed $\mathbf{GL}$-subvariety of $\mathbf{A}^{\lambda}$ with $\delta_Z < \delta_X$. Thus, by the inductive hypothesis, it is $\mathbf{GL}$-noetherian.
- We have $X = Z \cup U$. Since Z and U are both $\mathbf{GL}$-noetherian, so is X by Exercise 14.1(b).

We make some comments on the general case:

- One shows that $\mathbf{A}^{\underline{\lambda}}$ is $\mathbf{GL}$-noetherian by induction on the size of $\underline{\lambda}$ (in a certain sense).
- Write $\underline{\lambda} = \underline{\mu} \cup \nu$ where ν is a partition in $\underline{\lambda}$ of maximal size. By the inductive hypothesis, $\mathbf{A}^{\underline{\mu}}$ is $\mathbf{GL}$-noetherian.
- Let $\pi \colon \mathbf{A}^{\underline{\lambda}} \to \mathbf{A}^{\underline{\mu}}$ be the projection map. We show that $\pi^{-1}(Z)$ is $\mathbf{GL}$-noetherian for $Z \subset \mathbf{A}^{\underline{\mu}}$ a closed $\mathbf{GL}$-subvariety, proceeding by noetherian induction on Z. In other words, if we fix Z we can assume that for any proper closed $Z' \subset Z$ we already know that $\pi^{-1}(Z')$ is $\mathbf{GL}$-noetherian.
- Fix Z. Given $X \subset \pi^{-1}(Z)$, we define δ_X to be the smallest degree of an element of the ideal of X inside of the coordinate ring of $\pi^{-1}(Z)$. One shows that X is $\mathbf{GL}$-noetherian by induction on δ_X, similar to the previous argument. □

REMARK 14.4. We expect that $\mathbf{GL}$-varieties should satisfy stronger noetherian conditions. For example, ACC should hold for all closed $\mathbf{GL}$-subschemes (even non-reduced ones). This is not known yet though.

Exercises

Exercise 14.1. We establish some basic properties of equivariant noetherianity. In what follows, X is a space on which G acts, and all subsets are endowed with the subspace topology.

(a) Suppose X is G-noetherian and A is a G-stable subset. Show that A is G-noetherian.
(b) Suppose $X = A \cup B$ where A and B are G-stable and G-noetherian. Show that X is G-noetherian.
(c) Suppose H is a subgroup of G and Y is an H-stable subset of X that is H-noetherian. Show that $\bigcup_{g \in G} gY$ is G-noetherian.

Exercise 14.2. Let X be a closed $\mathbf{GL}$-subvariety of $\mathbf{A}^{\underline{\lambda}}$. Show that there are finitely many functions $f_1, \ldots, f_r$ on $\mathbf{A}^{\underline{\lambda}}$ such that X is the common zero locus of the $\mathbf{GL}$-orbits of the f_i's.

Additional exercises

Exercise 14.3. Let R be the coordinate ring of $\mathbf{A}^{(1)} \times \mathbf{A}^{(1)}$. Show that $\mathbf{GL}$-ideals in R satisfy ACC.

Exercise 14.4. Let $\mathfrak{S}$ be the infinite symmetric group (whichever version you prefer), let $\mathfrak{S}$ act on $R = \mathbf{C}[x_1, x_2, \ldots]$ by permuting variables, and let $X = \operatorname{Spec}(R)$. Show that X is $\mathfrak{S}$-noetherian. [Warning: I do not know an easy proof!]

Notes

Draisma proved his theorem in [Dr]. That such a statement might be true seems to be first suggested in [Sn, §6]. The case of $\mathbf{A}^{(3)}$ was treated earlier by Derksen, Eggermont, and Snowden [DES], following prior work of Eggermont [Eg] in the degree two case.