Lecture 15

Stillman's conjecture via GL-varieties

In this lecture, we sketch a geometric proof of Stillman's conjecture based on Draisma's theorem. For a k-algebra A, let $\mathbf{R}_A$ be the inverse limit of the rings $A[x_1, \ldots, x_n]$ in the category of graded rings, where A is concentrated in degree 0. If $A \to B$ is a homomorphism of k-algebras, there is an induced homomorphism $\mathbf{R}_A \to \mathbf{R}_B$. In particular, if M is a graded $\mathbf{R}_A$-module and x is a point of $\mathrm{Spec}(A)$ then $M_x = M \otimes_A \kappa(x)$ is naturally a graded $\mathbf{R}_{\kappa(x)}$-module. Thus M gives rise to a family of $\mathbf{R}$-modules over $\mathrm{Spec}(A)$. We let $p_M(x)$ be the projective dimension of M_x as an $\mathbf{R}_{\kappa(x)}$-module.

Proposition 15.1. *Let M be a finitely presented $\mathbf{R}_A$-module, and suppose A is an integral domain. Then there is a dense open set U of $\mathrm{Spec}(A)$ such that $p_M(x)$ is constant for $x \in U$.*

Proof. Let $K = \mathrm{Frac}(A)$. The ring $K \otimes_A \mathbf{R}_A$ is a polynomial ring (Exercise 15.3). It follows that $K \otimes_A M$ has finite projective dimension over $K \otimes_A \mathbf{R}_A$ (Exercise 15.1). Now one "spreads out" the resolution over $\mathrm{Spec}(A)$; see Exercise 15.5 for how this works in a simpler case. □

Fix positive integers $d_1, \ldots, d_r$. Let A be the $\mathbf{GL}$-algebra

$$A = \mathrm{Sym}(\mathrm{Sym}^{d_1}(\mathbf{V}) \oplus \cdots \oplus \mathrm{Sym}^{d_r}(\mathbf{V})).$$

Note that $\mathrm{Spec}(A)$ is exactly the $\mathbf{GL}$-variety $\mathbf{A}^{\underline{\lambda}}$ considered in Example 12.4. Explicitly, A is the polynomial ring in variables $c_{i,\alpha}$, where $1 \le i \le r$, and α varies over exponent vectors of degree d_i (so that x^α varies over degree d_i monomials). Define F_i to be the element of $\mathbf{R}_A$ given by $F_i = \sum_{|\alpha|=d_i} c_{i,\alpha} x^\alpha$. Thus $(F_1, \ldots, F_r)$ is a universal tuple of degree $(d_1, \ldots, d_r)$, in that any such tuple in $\mathbf{R}$ can be obtained from this one along a base change $A \to k$. Let $M = \mathbf{R}_A/(F_1, \ldots, F_r)$.

Theorem 15.2. *The space $\mathrm{Spec}(A)$ admits a finite decomposition $\bigsqcup_{i=1}^n K_i$ where each K_i is a locally closed $\mathbf{GL}$-subvariety of $\mathrm{Spec}(A)$ such that p_M is constant on each K_i.*

Proof. Let $Z = \mathrm{Spec}(A')$ be a closed $\mathbf{GL}$-subvariety of $\mathrm{Spec}(A)$. Applying Proposition 15.1 to $M \otimes_A A'$, we see that there is a dense open subset V_0 of Z such that p_M is constant on V_0. For $g \in \mathbf{GL}$, the modules M_x and M_{gx} have the same projective dimensions; in other words, the function p_M is $\mathbf{GL}$-invariant. Thus, putting $U = \bigcup_{g \in \mathbf{GL}} gV_0$, we see that p_M is constant on V. The set V is an open dense $\mathbf{GL}$-stable subset of Z.

The result now follows Draisma's theorem (Theorem 14.3). Indeed, let $Z_0 = \mathrm{Spec}(A)$. By the previous paragraph, there is a dense open $K_0 \subset Z_0$ such that p_M

is constant on K_0. Now put $Z_1 = Z_0 \setminus K_0$, which is a closed $\mathbf{GL}$-subvariety of Z_0. Applying the previous paragraph again, there is a dense open $K_1 \subset Z_1$ such that p_M is constant on K_1. Now put $Z_2 = Z_1 \setminus K_1$, and continue. The descending chain $Z_\bullet$ of closed $\mathbf{GL}$-subvarieties of $\mathrm{Spec}(A)$ stabilizes by Draisma's theorem, and it must stabilize at the empty set. □

COROLLARY 15.3. *There is an integer $N = N(d_1, \ldots, d_r)$ with the following property: if $f_1, \ldots, f_r \in \mathbf{R}$ have degrees $d_1, \ldots, d_r$ then the projective dimension of $\mathbf{R}/(f_1, \ldots, f_r)$ is at most N.*

Proof. This follows from the theorem since $\mathbf{R}/(f_1, \ldots, f_r)$ has the form M_x for some $x \in \mathrm{Spec}(A)$. □

Stillman's conjecture is in fact a special case of the above corollary:

COROLLARY 15.4. *If $f_1, \ldots, f_r$ are homogeneous elements of $R_n = k[x_1, \ldots, x_n]$ of degrees $d_1, \ldots, d_r$ then the projective dimension of $R_n/(f_1, \ldots, f_r)$ is at most the number N from Corollary 15.3.*

Proof. See Exercise 15.5. □

Exercises

Exercise 15.1. Let $\mathfrak{a}$ be a finitely generated ideal of polynomial ring (such as $\mathbf{R}$ or $K \otimes_A \mathbf{R}$). Show that $\mathrm{pdim}_{\mathbf{R}}(\mathfrak{a})$ is finite. [This is elementary: you don't need to use anything like Stillman's conjecture.]

Let A be an integral k-algebra with fraction field K.

Exercise 15.2. How are the rings $\mathbf{R}_K$ and $K \otimes_A \mathbf{R}_A$ related? [Is there a homomorphism? Is it injective/surjective/isomorphism?]

Exercise 15.3. Show that $K \otimes_A \mathbf{R}_A$ is a polynomial K-algebra. [Hint: use Theorem 4.3].

Additional exercises

Exercise 15.4. Regard the polynomial algebra $A[x_1, \ldots, x_n]$ as a graded ring where A has degree 0 and each x_i has degree 1. Suppose that M is a finitely presented graded A-module. Show that there is an open dense subset U of $\mathrm{Spec}(A)$ such that the Betti table of M_x is constant for $x \in U$.

[Hint: consider the resolution of $K \otimes_A M$ over $K[x_1, \ldots, x_n]$. The differentials are matrices with entires in K. They therefore belong to $A[1/f]$ for some non-zero $f \in A$. In this way, one can spread out the complex to $A[1/f]$. Further argument is needed to show that one can choose f so that the complex is exact, and that the projective dimension doesn't go down.]

Exercise 15.5. Deduce Corollary 15.4 from Corollary 15.3.

Notes

The proof in this lecture is based on the material in [ESS2, §5]. All of the technical details are treated there. For a more expository account (omitting many details), see [ESS3, §9].