Lecture 15

Stillman’s conjecture via GL-varieties

In this lecture, we sketch a geometric proof of Stillman’s conjecture based on Draisma’s theorem. For a $k$-algebra $A$, let $R_A$ be the inverse limit of the rings $A[x_1,\ldots,x_n]$ in the category of graded rings, where $A$ is concentrated in degree 0. If $A \to B$ is a homomorphism of $k$-algebras, there is an induced homomorphism $R_A \to R_B$. In particular, if $M$ is a graded $R_A$-module and $x$ is a point of $\text{Spec}(A)$ then $M_x = M \otimes_A \kappa(x)$ is naturally a graded $R_{\kappa(x)}$-module. Thus $M$ gives rise to a family of $R$-modules over $\text{Spec}(A)$. We let $p_M(x)$ be the projective dimension of $M_x$ as an $R_{\kappa(x)}$-module.

**Proposition 15.1.** Let $M$ be a finitely presented $R_A$-module, and suppose $A$ is an integral domain. Then there is a dense open set $U$ of $\text{Spec}(A)$ such that $p_M(x)$ is constant for $x \in U$.

**Proof.** Let $K = \text{Frac}(A)$. The ring $K \otimes_A R_A$ is a polynomial ring (Exercise 15.3). It follows that $K \otimes_A M$ has finite projective dimension over $K \otimes_A R_A$ (Exercise 15.1). Now one “spreads out” the resolution over $\text{Spec}(A)$; see Exercise 15.5 for how this works in a simpler case.

Fix positive integers $d_1,\ldots,d_r$. Let $A$ be the $\text{GL}$-algebra

$$A = \text{Sym}(\text{Sym}^{d_1}(V) \oplus \cdots \oplus \text{Sym}^{d_r}(V)).$$

Note that $\text{Spec}(A)$ is exactly the $\text{GL}$-variety $A^\Delta$ considered in Example 12.4. Explicitly, $A$ is the polynomial ring in variables $c_{i,\alpha}$, where $1 \leq i \leq r$, and $\alpha$ varies over exponent vectors of degree $d_i$ (so that $x^\alpha$ varies over degree $d_i$ monomials). Define $F_i$ to be the element of $R_A$ given by $F_i = \sum_{|\alpha|=d_i} c_{i,\alpha} x^\alpha$. Thus $(F_1,\ldots,F_r)$ is a universal tuple of degree $(d_1,\ldots,d_r)$, in that any such tuple in $R$ can be obtained from this one along a base change $A \to k$. Let $M = R_A/(F_1,\ldots,F_r)$.

**Theorem 15.2.** The space $\text{Spec}(A)$ admits a finite decomposition $\bigcup_{i=1}^n K_i$ where each $K_i$ is a locally closed $\text{GL}$-subvariety of $\text{Spec}(A)$ such that $p_M$ is constant on each $K_i$.

**Proof.** Let $Z = \text{Spec}(A')$ be a closed $\text{GL}$-subvariety of $\text{Spec}(A)$. Applying Proposition 15.1 to $M \otimes_A A'$, we see that there is a dense open subset $V_0$ of $Z$ such that $p_M$ is constant on $V_0$. For $g \in \text{GL}$, the modules $M_x$ and $M_{gx}$ have the same projective dimensions; in other words, the function $p_M$ is $\text{GL}$-invariant. Thus, putting $U = \bigcup_{g \in \text{GL}} gV_0$, we see that $p_M$ is constant on $V$. The set $V$ is an open dense $\text{GL}$-stable subset of $Z$.

The result now follows Draisma’s theorem (Theorem 14.3). Indeed, let $Z_0 = \text{Spec}(A)$. By the previous paragraph, there is a dense open $K_0 \subset Z_0$ such that $p_M$
is constant on $K_0$. Now put $Z_1 = Z_0 \setminus K_0$, which is a closed $\text{GL}$-subvariety of $Z_0$. Applying the previous paragraph again, there is a dense open $K_1 \subset Z_1$ such that $p_M$ is constant on $K_1$. Now put $Z_2 = Z_1 \setminus K_1$, and continue. The descending chain $Z_\bullet$ of closed $\text{GL}$-subvarieties of $\text{Spec}(A)$ stabilizes by Draisma’s theorem, and it must stabilize at the empty set.

**Corollary 15.3.** There is an integer $N = N(d_1, \ldots, d_r)$ with the following property: if $f_1, \ldots, f_r \in R$ have degrees $d_1, \ldots, d_r$ then the projective dimension of $R/(f_1, \ldots, f_r)$ is at most $N$.

**Proof.** This follows from the theorem since $R/(f_1, \ldots, f_r)$ has the form $M_x$ for some $x \in \text{Spec}(A)$.

Stillman’s conjecture is in fact a special case of the above corollary:

**Corollary 15.4.** If $f_1, \ldots, f_r$ are homogeneous elements of $R_n = k[x_1, \ldots, x_n]$ of degrees $d_1, \ldots, d_r$ then the projective dimension of $R_n/(f_1, \ldots, f_r)$ is at most the number $N$ from Corollary 15.3.

**Proof.** See Exercise 15.5.

**Exercises**

**Exercise 15.1.** Let $a$ be a finitely generated ideal of polynomial ring (such as $R$ or $K \otimes_A R$). Show that $\text{pdim}_R(a)$ is finite. [This is elementary: you don’t need to use anything like Stillman’s conjecture.]

Let $A$ be an integral $k$-algebra with fraction field $K$.

**Exercise 15.2.** How are the rings $R_K$ and $K \otimes_A R_A$ related? [Is there a homomorphism? Is it injective/surjective/isomorphism?]

**Exercise 15.3.** Show that $K \otimes_A R_A$ is a polynomial $K$-algebra. [Hint: use Theorem 4.3].

**Additional exercises**

**Exercise 15.4.** Regard the polynomial algebra $A[x_1, \ldots, x_n]$ as a graded ring where $A$ has degree 0 and each $x_i$ has degree 1. Suppose that $M$ is a finitely presented graded $A$-module. Show that there is an open dense subset $U$ of $\text{Spec}(A)$ such that the Betti table of $M_x$ is constant for $x \in U$.

[Hint: consider the resolution of $K \otimes_A M$ over $K[x_1, \ldots, x_n]$. The differentials are matrices with entries in $K$. They therefore belong to $A[1/f]$ for some non-zero $f \in A$. In this way, one can spread out the complex to $A[1/f]$. Further argument is needed to show that one can choose $f$ so that the complex is exact, and that the projective dimension doesn’t go down.]

**Exercise 15.5.** Deduce Corollary 15.4 from Corollary 15.3.
Notes

The proof in this lecture is based on the material in [ESS2, §5]. All of the technical details are treated there. For a more expository account (omitting many details), see [ESS3, §9].