

Beyond Stillman's conjecture

Stillman's conjecture asserts that the invariant “projective dimension” is bounded for ideals in the “Stillman regime,” where the number and degrees of generators is fixed. In this lecture, we will see that more general invariants of ideals are also bounded in the Stillman regime.

An *ideal invariant* is a rule ν assigning a quantity $\nu(I) \in \mathbf{N} \cup \{\infty\}$ to each homogeneous ideal I in a standard-graded polynomial ring $R = k[x_1, \dots, x_n]$ such that $\nu(I)$ only depends on the pair (R, I) up to isomorphism. In this context, an isomorphism $(R, I) \rightarrow (R', I')$ is an isomorphism $f: R \rightarrow R'$ of graded rings such that $f(I) = I'$; essentially, this just allows for linear changes in the variables. There are many examples of ideal invariants: projective dimension, regularity, the (i, j) Betti number, etc.

We say that an ideal invariant ν is *Stillman bounded* if for each tuple $\mathbf{d} = (d_1, \dots, d_r)$ of positive integers there is a quantity $C = C(\mathbf{d})$ such that whenever I is generated by r elements of degrees d_1, \dots, d_r we have $\nu(I) \leq C$ or $\nu(I) = \infty$.

It is hard to say anything meaningful about general ideal invariants. We will require two conditions on our ideal invariant ν to get some control. The first condition is fairly straightforward: we say that ν is *cone-stable* if $\nu(I) = \nu(I[x])$ for all (R, I) ; that is, adjoining a new variable does not change the invariant. Projective dimension is easily seen to be cone-stable.

The second condition is that $\nu(I)$ should be a continuous function of I , in a certain sense. This is a bit technical to formulate. We assume k is algebraically closed for simplicity. Let B be a finitely generated k -algebra and let $I \subset B[x_1, \dots, x_n]$ be a homogeneous ideal (with B in degree 0). Given a (closed) point s of $X = \text{Spec}(B)$, i.e., a k -algebra homomorphism $s: B \rightarrow k$, let I_s be the extension of I along the homomorphism $B[x_1, \dots, x_n] \rightarrow k[x_1, \dots, x_n]$. We thus have a function

$$(\text{closed points of } X) \rightarrow \mathbf{N} \cup \{\infty\}, \quad s \mapsto \nu(I_s).$$

We say that ν is *strongly upper semi-continuous* if this function is always upper semi-continuous. (Recall that “upper semi-continuous” means that the function can “jump up” as a limit point is approached.) This condition is quite strong, as most invariants only behave well in flat families. We say therefore say that ν is *weakly upper semi-continuous* if the above function is upper semi-continuous whenever $B[x_i]/I$ is flat as a B -module. Projective dimension is weakly upper semi-continuous: in fact, it is constant in flat families.

The following is the main theorem:

THEOREM 16.1. *Any ideal invariant that is cone-stable and weakly upper semi-continuous is Stillman bounded.*

Proof. We simply sketch a proof. Let ν be a cone-stable ideal invariant and fix a tuple $\mathbf{d} = (d_1, \dots, d_r)$ of positive integers.

Let $X_{\mathbf{d},n} = \text{Sym}^{d_1}(k^n) \times \dots \times \text{Sym}^{d_r}(k^n)$, regarded as a finite-dimensional affine variety. We have a natural inclusions $X_{\mathbf{d},n} \subset X_{\mathbf{d},n+1}$, and we let $X_{\mathbf{d}}$ be the direct limit. This is an ind-scheme (and not a scheme). The infinite general linear group \mathbf{GL} naturally acts on $X_{\mathbf{d}}$. Draisma's theorem still holds in this context, though it requires non-trivial justification.

Each (closed) point s of $X_{\mathbf{d}}$ defines a finitely homogeneous generated ideal I_s of $k[x_1, x_2, \dots]$. Since ν is cone-stable, it is well-defined on such ideals (just go to a finite variable ring where all the generators live). We therefore have a well-defined function

$$X_{\mathbf{d}} \rightarrow \mathbf{N} \cup \{\infty\}, \quad s \mapsto \nu(I_s)$$

Let $Z_n \subset X_{\mathbf{d}}$ be the set of points s such that $\nu(I_s) \geq n$. The Z_n 's form a descending chain of subsets of $X_{\mathbf{d}}$.

Suppose that ν is strongly upper semi-continuous. This exactly means that the Z_n 's are closed. By Draisma's theorem, the Z_n 's stabilize. If $Z_n = Z_{n_0}$ for $n > n_0$ then we must have $\nu(I_s) \leq n_0$ or $\nu(I_s) = \infty$ for all s , and so ν is Stillman bounded.

Now suppose that ν is only weakly upper semi-continuous. Using the usual form of Stillman's conjecture, one shows that there is a finite locally closed stratification of $X_{\mathbf{d}}$ such that the family $\{k[x_i]/I_s\}$ is flat on each stratum. One can then argue as in the previous paragraph. \square

While Theorem 16.1 appears to be quite general, it is actually rather difficult to find examples of ideal invariants to which it applies. Here's one:

EXAMPLE 16.2. Fix an integer $c \geq 1$. Given a homogeneous ideal $I \subset k[x_1, \dots, x_n]$, consider the space X_I of codimension c linear of \mathbf{A}^n contained in $V(I)$; this is naturally a scheme, in fact, a closed subscheme of the appropriate Grassmannian. We define $\nu(I)$ as follows: if X_I is a finite collection of reduced points then $\nu(I) = \#X_I$; otherwise $\nu(I) = \infty$. One can use the theorem to show that ν is Stillman bounded (though it is not quite cone-stable). This example is inspired by the classical fact that a smooth cubic surface has 27 lines.

REMARK 16.3. Let $Y_{\mathbf{d}}$ be the set of isomorphism classes of homogeneous ideals of $k[x_1, x_2, \dots]$ generated by r elements of degrees d_1, \dots, d_r . There is a surjective \mathbf{GL} -invariant map $X_{\mathbf{d}} \rightarrow Y_{\mathbf{d}}$. Giving $Y_{\mathbf{d}}$ the quotient topology, it follows from Draisma's theorem that $Y_{\mathbf{d}}$ is a noetherian topological space.

Exercises

Exercise 16.1. Give an example of an ideal invariant that is not Stillman bounded.

Exercise 16.2. Verify directly that projective dimension is cone-stable and weakly upper semi-continuous.

Notes

This lecture is based on [ESS1]. For ideal invariants that are strongly upper semi-continuous, one can improve Theorem 16.1; see [ESS1, §5.2].