

Ultrahomogeneous forms

In what we have done so far, all infinite strength elements of \mathbf{R} have been more or less equal: for example, all obey Principle 1.7 and are all \mathbf{GL} -generic (see Exercise 12.3). It is therefore reasonable to ask: are there any interesting features that some infinite strength forms have that others do not? In this lecture, we see that the answer is a resounding yes!

To motivate our discussion, let us first return to the finite variable setting. All non-degenerate quadratic forms in n -variables over the complex numbers are equivalent, i.e., belong to the same \mathbf{GL}_n -orbit. There is no statement like this for cubic forms: cubic forms in n variables are a zoo, with ever more creatures as n grows. One might therefore think the infinite dimensional case would be hopeless.

There is a roughly similar situation with graphs, in that graphs are also a zoo that becomes increasing complex as the number of vertices increases. However, it turns out that, rather amazingly, there is a “best” infinite graph:

THEOREM 17.1. *There exists a countable graph Γ , called the Rado graph, having the following two properties:*

- (a) *Universality: every finite graph occurs as an induced subgraph of Γ .*
- (b) *Ultrahomogeneity: any isomorphism $\Delta \rightarrow \Delta'$ of finite induced subgraphs of Γ extends to an automorphism of Γ .*

Moreover, Γ is unique up to isomorphism.

The ultrahomogeneity condition implies that the automorphism group G of the Rado graph is quite large. Indeed, let V be the vertex set of the Rado graph, and write $V^{(n)}$ for the set of n -element subsets of V . Then the orbits of G on $V^{(n)}$ are in bijective correspondence with isomorphism classes of graphs with n vertices (Exercise 17.2). In particular, G has finitely many orbits on $V^{(n)}$. A permutation group with this property is called *oligomorphic*.

It turns out that there is an analog of Theorem 17.1 for forms. To state this, it will be convenient to introduce some terminology. A *symmetric d -form* on a vector space V is a linear map $\text{Sym}^d(V) \rightarrow \mathbf{C}$. A *symmetric d -space* is a vector space equipped with a symmetric d -form. For example, a symmetric 2-space is a quadratic space, i.e., a vector space with a quadratic form. There are obvious notions of isomorphism and embedding for symmetric d -spaces. We can now state our theorem:

THEOREM 17.2. *Fix $d \geq 1$. There exists a symmetric d -space V of countable dimension having the following two properties:*

- (a) *Universality: every finite dimensional symmetric d -space embeds into V .*

- (b) *Ultrahomogeneity*: if W and W' are finite dimensional subspaces of V equipped with the induced forms, then any isomorphism $W \rightarrow W'$ extends to an automorphism of V .

Moreover, V is unique up to isomorphism.

In other words, of the infinite strength degree d elements of \mathbf{R} , there is a unique isomorphism class of ultrahomogeneous forms; simply put, there is a “best” degree d form! As with the Rado graph, the automorphism group of this form is huge.

The above theorems can be proved using a construction called the *Fraïssé limit*. We provide a very abridged account. A *finite relational structure* is a finite set equipped with a number of relations (of possibly varying arities); for example, one can think of a finite graph as a finite relational structure with a single binary relation. Let \mathcal{C} be a class of finite relational structures. We consider the following two conditions:

- (JEP) We say that \mathcal{C} has the *joint embedding property* if any two members of \mathcal{C} embed into some other member.
- (AP) We say that \mathcal{C} has the *amalgamation property* if, given embeddings $W \rightarrow X$ and $W \rightarrow Y$ in \mathcal{C} , one can find a commutative square

$$\begin{array}{ccc} X & \longrightarrow & Z \\ \uparrow & & \uparrow \\ W & \longrightarrow & Y \end{array}$$

of embeddings in \mathcal{C} .

Notice that if \mathcal{C} has an initial member then (JEP) is a special case of (AP). We can now state Fraïssé’s theorem:

THEOREM 17.3. *Suppose that \mathcal{C} satisfies (AP) and (JEP) and has countably many members (up to isomorphism). Then there is a countable structure, constructed as a union of a chain in \mathcal{C} , that is universal (all elements of \mathcal{C} embed into it) and ultrahomogeneous. It is unique, up to isomorphism.*

It is easy to deduce Theorem 17.1 from Theorem 17.3. It is also not too hard to deduce Theorem 17.2 from Theorem 17.3 when the coefficient field is countable. In general, one needs a more powerful variant of Theorem 17.3.

Exercises

Exercise 17.1. Let \mathcal{C} be the class of finite graphs. Show that \mathcal{C} satisfies (AP). [Note that the empty graph is initial, and so (JEP) holds as well.]

Exercise 17.2. Let G be the automorphism group of the Rado graph, let V be the vertex set of the Rado graph, and let $V^{(n)}$ be the set of n -element subsets of V . Show that the G -orbits on $V^{(n)}$ are in natural bijection with isomorphism classes of graphs on n vertices.

Exercise 17.3. Show that $(\mathbf{Q}, <)$ is ultrahomogeneous, as a totally ordered set. It is thus the Fraïssé limit of the class of finite totally ordered sets.

Exercise 17.4. Work over a finite field \mathbf{F} of characteristic $\neq 2, 3$. Explain how a cubic space can be encoded as a relational structure. Let \mathcal{C} be the class of finite dimensional cubic spaces over \mathbf{F} . Show that \mathcal{C} satisfies (AP). [Note that the zero space is initial, and so (JEP) holds as well.]

Notes

Theorem [17.2](#), and some generalizations and related material, will appear in [\[HS\]](#). For background on Fraïssé limits, I highly recommend Cameron's book [\[Cam\]](#).