LECTURE 18

Universality of strength

In the previous lecture, we saw that there is an essentially unique degree d form that is both ultrahomogeneous and universal. One might therefore wonder about forms having just one of these properties. In this lecture, we consider the universal case. The main result is the following theorem:

THEOREM 18.1. *Let V be a symmetric d-space of countable infinite dimension for which the defining form has infinite strength. Then V is universal, i.e., any finite dimensional symmetric d-space embeds into V.*

This theorem can be reformulated in the language of forms as follows:

THEOREM 18.2. *Let f be a homogeneous element of $\mathbf{R}$ of degree d and infinite strength. Given any n and any homogeneous g of R_n of degree d, there exists a continuous homomorphism $\varphi\colon \mathbf{R} \to R_n$ such that $\varphi(f) = g$.*

In the above theorem, "continuous" is taken with respect to the inverse limit topology on $\mathbf{R}$. Concretely, this simply means that $\varphi(x_i) = 0$ for all but finitely many i, and that φ commutes with infinite sums.

This theorem was first proved by Kazhdan–Ziegler [KaZ]. It was then generalized to arbitary polynomial functors by Bik, Danelon, Draisma, and Eggermont [BDDE]. The theorem is deduced from a more general result, which we now explain.

Let $X = \operatorname{Spec}(R)$ be a $\mathbf{GL}$-variety. By definition, R is a polyomial representation of $\mathbf{GL}$, which means that it decomposes into a sum of Schur functors $\mathbf{S}_{\underline{\lambda}}(\mathbf{V})$. However, since Schur functors are functors, *any* linear endomorphism of $\mathbf{V}$ induces an endomorphism of $\mathbf{S}_{\underline{\lambda}}(\mathbf{V})$. It follows that the monoid $\operatorname{End}(\mathbf{V})$ acts on R, and therefore on X. The following is the more general result:

THEOREM 18.3. *Let S be a* subset *of (the $\mathbf{C}$-points of) $\mathbf{A}^{(d)}$ that is closed under the action of* $\operatorname{End}(\mathbf{V})$. *Then exactly one of the following two possibilites holds:*

(a) S is contained in the strength $\le s$ locus, for some s.
(b) S contains every degree d polynomial.

In the above theorem, we identify $\mathbf{A}^{(d)}$ with the degree d part of the inverse limit ring $\mathbf{R}$. Thus a point in $\mathbf{A}^{(d)}$ is a formal linear combination of degree d monomials in the variables $\{x_i\}_{i\ge 1}$. We say that a point of $\mathbf{A}^{(d)}$ is a polynomial if it is a finite sum of monomials.

We now explain how to obtain Theorem 18.2 from Theorem 18.3. Let f be a degree d element of $\mathbf{R}$ of infinite strength. Let $X \subset \mathbf{A}^{(d)}$ be the $\operatorname{End}(\mathbf{V})$-orbit of f, i.e., $S = \{\sigma \cdot f \mid \sigma \in \operatorname{End}(\mathbf{V})\}$. Clearly, S is not contained in any strength $\le s$ locus, and so S contains every degree d polynomial by Theorem 18.3. Let $g \in R_n$

be of degree d. We can then find $\sigma \in \mathrm{End}(\mathbf{V})$ such that $\sigma \cdot f = g$. Using σ, one produces a continuous homomorphism $\varphi \colon \mathbf{R} \to R_n$ such that $\varphi(f) = g$. This proves Theorem 18.2.

Exercises

There are no exercises for the final lecture!

Notes

Theorem 18.2 was proved for $d = 3$ by Derksen, Eggermont, and Snowden [DES], and in general by Kazhdan and Ziegler [KaZ]. The statement was generalized to arbitrary polynomial functors by Bik, Danelon, Draisma, and Eggermont [BDDE]..