Lecture 1

Stillman's conjecture

Let k be a field and let $R_n = k[x_1, \dots, x_n]$ be the n-variable polynomial ring. Recall the famous Hilbert syzygy theorem:

Theorem 1.1. *Any R_n-module has projective dimension $\leq n$.*

The R_n-module $k = R_n/(x_1, \dots, x_n)$ has projective dimension n, which shows that the theorem is optimal. More generally, if $f_1, \dots, f_r$ is a regular sequence in R_n then $R_n/(f_1, \dots, f_r)$ has projective dimension r. One might therefore expect that the projective dimension of an ideal generated by r elements would have projective dimension at most r. This is too simplistic, but only slightly: Mike Stillman formulated the following conjecture around the year 2000.

Conjecture 1.2. *The projective dimension of a homogeneous ideal I of R_n can be bounded in terms of the number and degrees of generators I (and independent of n). In other words, given $d_1, \dots, d_r$ there is $N = N(d_1, \dots, d_r)$ such that $\mathrm{pdim}_{R_n}(I) \leq N$ whenever $I \subset R_n$ is generated by homogeneous polynomials $f_1, \dots, f_r$ of degrees $d_1, \dots, d_r$.*

Remark 1.3. For any n, there are examples where $R/(f_1, f_2, f_3)$ has projective dimension n. Thus the dependence on degree is necessary in Stillman's conjecture. Also, for fixed d and $n \gg d$, there are examples of $R/(f_1, f_2, f_3)$ of projective dimension at least $d^{(\sqrt{d}-1)/2}$. Thus the bound N in the conjecture must be rather fast-growing.

Thie conjecture is now a theorem, first proved by Ananyan and Hochster in 2016. We'll give two proofs of the conjecture this week following the paper of Erman, Sam, and Snowden. In the rest of this lecture, we'll introduce some key concepts. The most important of these is the following definition, due to Ananyan–Hochster:

Definition 1.4. A homogeneous element f of a graded ring has *strength* $\leq s$ if there is an expression $f = \sum_{i=1}^{s} g_i h_i$ where g_i and h_i are homogeneous elements of positive degree[1]. If there is no such expression for any s then f has infinite strength.

Example 1.5. A polynomial has strength 0 if and only if it is zero, and strength 1 if and only if it is non-zero and not irreducible. The polynomial $x_1^2 + x_2^2$ clearly has strength ≤ 2, but it has strength 1 if $-1 = i^2$ is a square in k, since

$$x_1^2 + x_2^2 = (x_1 + ix_2)(x_1 - ix_2).$$

The following mild generalization of strength will also be helpful:

[1]This definition is off by one from the original definition of Ananyan–Hochster.

DEFINITION 1.6. Let $\{f_i\}_{i\in I}$ be a family of homogeneous elements in a graded k-algebra. The *collective strength* of the f_i's is the minimal strength of a non-trivial homogeneous k-linear combination of the f_i's. (Note that f_i's of different degree do not interact in this definition.)

The proof of Ananyan–Hochster actually revealed the following general principle that can be used to predict statements similar to Stillman's conjecture, and will guide us towards the proof of Stillman's conjecture that we present:

PRINCIPLE 1.7. *If $f_1, \ldots, f_r$ are homogeneous polynomials of high collective strength then $f_1, \ldots, f_r$ behave approximately like r independent variables.*

We briefly explain how Stillman's conjecture is a consequence of an instance of this principle. Let $f_1, \ldots, f_r \in R_n$ be given, and let I be the ideal they generate. If $f_1, \ldots, f_r$ have sufficiently "large" collective strength, then they form a regular sequence (by the Principle 1.7), and so $\operatorname{pdim}_{R_n}(I) \leq r-1$. On the other hand, if $f_1, \ldots, f_r$ have "small" collective strength, some linear combination can be written in terms of lower degree polynomials. By repeating this procedure, we eventually express $f_1, \ldots, f_r$ in terms of polynomials $g_1, \ldots, g_s$ that have high collective strength. It is not difficult to bound s in terms of the degrees of the f_i's. The projective dimension of the ideal $J \subset k[g_1, \ldots, g_s]$ generated by the f_i's is $\leq s$ by Hilbert's theorem; using the fact that the g_i's form a regular sequence, this allows us to conclude $\operatorname{pdim}_{R_n}(I) \leq s$ as well.

Exercises

Exercise 1.1. Determine the strength of $x_1^2 + \cdots + x_n^2$ over the complex numbers.

Exercise 1.2. Let R be any graded k-algebra and let R_+ be the ideal of positive degree elements in R.

(a) Show that a homogeneous element has finite strength if and only if it belongs to R_+^2. In particular, the finite strength elements form an ideal.
(b) Show that a collection of homogeneous elements has infinite collective strength if and only if it maps to a k-linearly independent set in R_+/R_+^2.
(c) Show that the homogeneous elements of R of infinite strength generate R as a ring. Even better, show that is a family of elements $\{f_i\}_{i\in I}$ of infinite collective strength that generates R.

Additional exercises

Exercise 1.3. Let $f \in k[x_1, \ldots, x_n]$ be homogeneous of positive degree d with k a field of characteristic 0, and let J be the ideal generated by the partial derivatives of f.

(a) Prove *Euler's formula*: $f = \frac{1}{d}\sum_{i=1}^n x_i \frac{\partial f}{\partial x_i}$.
(b) If f has strength $\leq s$ show that J is contained in an ideal generated by $\leq 2s$ elements.

(c) Conversely, if J is contained in an ideal generated by $\leq s$ elements show that f has strength $\leq s$.

Exercise 1.4. Show that $k = R_n/(x_1, \dots, x_n)$ has projective dimension n as an R_n-module.

Exercise 1.5. Let $V_1, \dots, V_n$ be vector spaces. Let $x \in V_1 \otimes \cdots \otimes V_n$. Recall that x is a *pure tensor* if $x = v_1 \otimes \cdots \otimes v_n$ for some $(v_1, \dots, v_n) \in V_1 \times \cdots \times V_n$. We say that x has *tensor rank* $\leq r$ if x is a sum of $\leq r$ pure tensors.

(a) Show that the tensor rank of x is at most $\prod_{i=1}^n \dim V_i$, assuming each V_i is finite dimensional.
(b) Let V be finite dimensional. Recall there is a natural isomorphism $\mathrm{End}(V) = V \otimes V^*$. Show that the tensor rank of an element of $V \otimes V^*$ coincides with its usual rank as an endomorphism of V.
(c) Again, let V be finite dimensional, and suppose k has characteristic $\neq 2$. We can identify the space $\mathrm{Sym}^2(V)$ of degree 2 polynomials in V with a subspace of $V^{\otimes 2}$. How do strength and tensor rank compare?

Exercise 1.6. Show that the polynomial $\sum_{i=1}^n x_i y_i z_i$ has strength n. (This is not so easy; see [DES, §4] for a proof.)

Notes

Stillman posed his conjecture around the year 2000. It first appeared in print in [En]; see also [PS]. The conjecture was first proved by Ananyan and Hochster [AH] in 2016. Using ideas from this proof, Erman, Sam, and Snowden [ESS2] gave two new and simpler proofs in 2018. Shortly thereafter, Draisma, Lasoń, and Leykin [DLL] gave a fourth proof using related ideas (but with some new elements). An expository account of the work appears in [ESS3].