Lecture 2

Ultraproducts

Principle 1.7 is an asymptotic statement: as collective strength increases, polynomials behave more and more like independent variables. This suggests that we might be able to pass to a limiting situation where we can say that things of infinite collective strength behave exactly like independent variables. There are two ways to create a limit of polynomial rings that will be relevant to us: inverse limits and ultraproducts. Inverse limits are reasonably well-known. Ultraproducts, on the other hand, are a bit more obscure (and much more subtle). We therefore review the basics of ultraproducts in this lecture.

Let I be a set, let x be an element of I, and let $\mathcal{F} = \mathcal{F}_x$ be the collection of all subsets of I containing x. The collection $\mathcal{F}$ satisfies the following conditions:

(a) The empty set is not in $\mathcal{F}$.
(b) Given $A \subset B$ with $A \in \mathcal{F}$ we have $B \in \mathcal{F}$.
(c) If $A, B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$.
(d) If A is any subset of I then either A of $I \setminus A$ belongs to $\mathcal{F}$.

This motivates the following definition:

Definition 2.1. An *ultrafilter* on a set I is a collection $\mathcal{F}$ of subsets of I satisfying conditions (a)–(d) above.

Thus $\mathcal{F}_x$ is an example of an ultrafilter; an ultrafilter of this form is called *principal*. A natural question is if non-principal ultrafilters exist. This is answered in the following proposition:

Proposition 2.2. *A set I admits a non-principal ultrafilter if and only if it is infinite.*

Proof. Let I be an infinite set, and consider the ring $R = \prod_{i \in I} \mathbf{F}_2$, where $\mathbf{F}_2$ is the field with two elements. For a subset A of I, let 0_A be the element of R having 0's in the A coordinates and 1's in the remaining coordinates. We have a bijection

$$\{\text{ultrafilters on } I\} \leftrightarrow \{\text{maximal ideals of } R\}$$

as follows: the maximal ideal corresponding to the ultrafilter $\mathcal{F}$ is $\{0_A \mid A \in \mathcal{F}\}$. The proof is left as an exercise (Exercise 2.6).

Given an element $x \in I$ we have a ring homomorphism $\pi_x \colon R \to \mathbf{F}_2$ by projecting onto the x coordinate. The kernel $\mathfrak{m}_x$ of π_x consists of all elements whose x coordinate is 0. We thus see that $\mathfrak{m}_x$ corresponds to $\mathcal{F}_x$ under the above bijection. To construct a non-principal ultrafilter on I, it thus suffices to construct a maximal ideal of R that's not of the form $\mathfrak{m}_x$.

Let $\mathfrak{a} \subset R$ be the set of all elements whose coordinates have finitely many 1's. This is clearly an ideal of R. We have $\mathfrak{a} \not\subset \mathfrak{m}_x$ for any x, as $\mathfrak{a}$ contains the element that has a 1 in the x coordinate and 0 in all other coordinates. By Zorn's lemma, $\mathfrak{a}$ is contained in some maximal ideal $\mathfrak{m}$. Thus $\mathfrak{m}$ is a maximal ideal that is not one of the $\mathfrak{m}_x$'s, and therefore yields a non-principal ultrafilter.

Now suppose I is finite. We must show that every ultrafilter is principal. We leave this as an exercise (Exercise 2.3). $\square$

Fix a non-principal ultrafilter $\mathcal{F}$ on I. Let $\{X_i\}_{i \in I}$ be a family of sets. We define an equivalence relation $\sim$ on the product $\prod_{i \in I} X_i$ as follows: $x \sim y$ if there is a set $A \in \mathcal{F}$ such that $x_i = y_i$ for all $i \in A$. We now come to the main definition of this lecture:

Definition 2.3. The *ultraproduct* of the X_i's (with respect to $\mathcal{F}$) is $(\prod_{i \in I} X_i)/\sim$.

One can think of the ultraproduct X^* of the X_i's as a kind of limit of the X_i's. Given a sequence $(x_i)_{i \in I}$ with $x_i \in X_i$, there is tautologically a "limit" x^* in X^*: indeed, (x_i) is simply an element of the product, and thus its equivalence class is an element of X^*. Thus, in a sense, all sequences have a limit in the ultraproduct.

One very useful feature of ultraproducts is that if the X_i's have some kind of algebraic structure (group, ring, field, etc.) then X^* will naturally carry this structure as well; see Exercise 2.1.

Remark 2.4. Psychologically, it is useful to think of $\mathcal{F}$ as specifying a hypothetical point $*$ of I. The condition $x \sim y$ can then be thought of as saying that $x_i = y_i$ holds in a neighborhood of $*$.

Exercises

Exercise 2.1. Let I be a set equipped with a non-principal ultrafilter $\mathcal{F}$. Let $\{K_i\}_{i \in I}$ be a family of fields, and let K^* be their ultraproduct.

(a) Show that K^* is a field under coordinatewise operations. Pay special attention to reciprocals.
(b) Suppose that I is the set of prime numbers and $K_p = \mathbf{F}_p$. Show that K^* has characteristic 0.

Now suppose that for each $i \in I$ we have a K_i-vector space V_i. Let V^* be the ultraproduct of the V_i's.

(c) Show that V^* is naturally a K^*-vector space.
(d) Suppose $\dim_{K_i}(V_i) = n$ for all $i \in I$. Show that $\dim_{K^*}(V^*) = n$.

Exercise 2.2. Let I be a set equipped with a non-principal ultrafilter $\mathcal{F}$. Let $\mathbf{R}^*$ be the ultrapower of the real numbers $\mathbf{R}$. (Ultrapower is just the case where the X_i's are all the same set.) Identify $a \in \mathbf{R}$ with the element $(a, a, \ldots)$ in $\mathbf{R}^*$.

(a) Show that $\mathbf{R}^*$ carries a natural total order.

(b) Show that there is an element $\epsilon \in \mathbf{R}^*$ such that $0 < \epsilon < a$ for all positive $a \in \mathbf{R}$.

An element ϵ as in (b) is called an *infinitesimal*. These elements form the basis of non-standard analysis.

Additional exercises

Exercise 2.3. Show that any ultrafilter on a finite set is principal.

Exercise 2.4. Show that a non-principal ultrafilter contains all cofinite sets.

Exercise 2.5. Let I be the set of prime numbers, for $p \in I$ let $K_p = \mathbf{F}_p$, and let K^* be the ultraproduct of the K_p's with respect to an ultrafilter $\mathcal{F}$ on I.

(a) Show that one can choose $\mathcal{F}$ so that -1 is a square in K^*.
(b) Show that one can choose $\mathcal{F}$ so that -1 is not a square in K^*.
(c) Try to figure out various possibilities for the behavior of algebraic numbers in K^*. E.g., is it possible for K^* to contain no irrational algebraic numbers?

Exercise 2.6. Prove the correspondence between ultrafilters and maximal ideals from Proposition 2.2.

Notes

We proved Proposition 2.2 (existence of non-principal ultrafilters) using the axiom of choice (in the form of Zorn's lemma). As one might expect, Proposition 2.2 is independent of ZF set theory. In fact, Proposition 2.2 is significantly weaker than the axiom of choice: ZF together with Proposition 2.2 cannot even prove countable choice.