LECTURE 3

Limits of polynomial rings

As we said in the previous lecture, one might hope to pass to a limiting situation where Principle 1.7 becomes an exact statement. In this lecture, we introduce the two limiting objects that we'll use to realize this dream.

Recall that $R_n = k[x_1, \ldots, x_n]$. There is a natural map $R_{n+1} \to R_n$ that kills x_{n+1} and maps x_i to x_i for $1 \le i \le n$. Thus the R_n's form an inverse system. Our first limiting object is the following:

DEFINITION 3.1. The *inverse limit ring* $\mathbf{R}$ is the inverse limit of the rings R_n *in the category of graded rings*.

In the above definition, we have emphasized the inverse limit takes place in the category of graded rings. This means that $\mathbf{R}$ is itself a graded ring, and that its degree d piece $\mathbf{R}_d$ is the inverse limit of the degree d pieces of the R_n's. Explicitly, a homogeneous degree d element of $\mathbf{R}$ is a formal k-linear combination of degree d monomidals in the variables $\{x_i\}_{i \ge 1}$. For example, $\sum_{i \ge 1} x_i$ is a degree 1 element of $\mathbf{R}$ and $\sum_{i \ge 1} x_i^2$ is a degree 2 element.

In the polynomial ring R_n, the only elements of infinite strength are linear. However, the inverse limit ring $\mathbf{R}$ has many elements of infinite strength. For example, $\sum_{i \ge 1} x_i^n$ has infinite strength provided n is not divisible by the characteristic of k (Exercise 3.1). In particular, the limiting form of Principle 1.7 (which concerns elements of infinite collective strength) is an interesting statement in $\mathbf{R}$ (which we will consider in the subsequent lecture).

To prove Stillman's conjecture, we want to use the following style of argument: let $\{f_i\}$ be a sequence of polynomials of increasing strength; pass to a limit f that has infinite strength; use the limiting form of Principle 1.7 to say something about f; pass this information back to the f_i's. Unfortunately, if $\{f_i\}$ is an arbitrary sequence of polynomials, it won't necessarily have any kind of sensible limit in $\mathbf{R}$ (though see Exercise 3.3). However, we saw in the previous lecture that "all limits exist" in ultraproducts, so this suggests that we should be using an ultraproduct construction.

We now introduce the precise ultraproduct we want to use. Let $R_\infty = k[x_1, x_2, \ldots]$ be the polynomial ring in variables $\{x_i\}_{i \ge 1}$. Fix an infinite set I and a non-principal ultrafilter $\mathcal{F}$ on I.

DEFINITION 3.2. The *ultraproduct ring* $\mathbf{S}$ is the *graded* ultrapower of R_∞.

We need to clarify what the word "graded" is doing in the above definition. This means that $\mathbf{S}$ is a graded ring, and that its degree d piece $\mathbf{S}_d$ is the ultrapower of the degree d piece of R_∞. Note in particular that $\mathbf{S}_0 = k^*$ is the ultrapower of the coefficient field k, which will be much larger than k (unless k is finite).

Remark 3.3. One might wonder: why are we taking the ultrapower of R_∞ instead of taking the ultraproduct of the R_n's? In fact, it doesn't really matter which we use, but I find using R_∞ more natural. With the way we have set things up, I is an arbitrary index set, and any sequence of polynomials in the variables $\{x_i\}_{i \geq 1}$ indexed by I will have a limit in $\mathbf{S}$. If one wanted to use the ultraproduct of the family $\{R_n\}_{n \in \mathbf{N}}$, this would involve choosing an ultrafilter on $\mathbf{N}$, and one could only consider sequences $\{f_n\}$ where f_n uses the first n variables; this seems a bit arbitrary.

Exercises

Exercise 3.1. Work over the complex numbers, and let d be a positive integer. Show that $\sum_{i \geq 1} x_i^d$ has infinite strength in $\mathbf{R}$. [Hint: show that if this element had strength $s < \infty$ then any degree d polynomial in n variables would have strength $\leq s$ and obtain a contradiction.]

Exercise 3.2. Let $f_i \in k[x_1, x_2, \ldots]$ be homogeneous of degree d and let $f \in \mathbf{S}$ be the corresponding element. Show that f has finite strength if and only if there is some set J in the ultrafilter such that the elements $\{f_j\}_{j \in J}$ have bounded strength. Formulate and prove a similar result for collective strength of a finite collection of elements.

Additional exercises

Exercise 3.3. Suppose that k is a finite field. For each $n \geq 1$, let f_n be a homogeneous degree d polynomial in $k[x_1, \ldots, x_n]$.

(a) Show that there is a homogeneous degree d element f in $\mathbf{R}$ such that f maps to f_n for infinitely many n. [Hint: $\mathbf{R}_d$ is a compact topological space (how?).]
(b) Suppose that the strength of the f_n's goes to infinity. Show that f has infinite strength.
(c) Suppose that the f_n's have bounded strength. Show that f has finite strength.

Exercise 3.4. Suppose $\mathrm{char}(k) \neq 2$. Let $f = \sum_{1 \leq i \leq j} a_{i,j} x_i x_j$ be a degree two element of $\mathbf{R}$.

(a) Show that f has strength 1 if and only if $a_{i,j} a_{k,\ell} = a_{i,\ell} a_{j,k}$ (with the convention $a_{i,j} = a_{j,i}$).
(b) Show that the condition "f has strength $\leq s$" is equivalent to a system of polynomial equations in the coefficients $a_{i,j}$.

Exercise 3.5. Let $\mathbf{R}_{k^*}$ be the graded inverse limit of the rings $k^*[x_1, \ldots, x_n]$.

(a) Construct a natrual homomorphism of k^*-algebras $\varphi \colon \mathbf{S} \to \mathbf{R}_{k^*}$. [Hint: given $f \in \mathbf{S}$ define $\varphi(f)$ by specifying the coefficient of each monomial.]
(b) Determine if φ is injective or surjective (or both or neither).

Exercise 3.6. Write down an element of the inverse limit of the R_n's in the category of (ungraded) rings that does not belong to $\mathbf{R}$.

Notes

The rings **R** and **S** were introduced (in the context of Stillman's conjecture) in [ESS2, §5] and [ESS2, §4]. A more expository treatment of these objects can be found in [ESS3, §4] and [ESS2, §5].

We defined **S** as the graded ultrapower of $k[x_1, x_2, \ldots]$. More generally, one could start with a family of fields $\{k_i\}_{i \in I}$ and define **S** as the graded ultraproduct of the rings $k_i[x_1, x_2, \ldots]$. This allows one to prove statements (like Stillman's conjecture) uniformly in the coefficient field. We have opted for the less general set-up in these lectures just to keep things a bit more simple.