Lecture 4

Big polynomial rings

Principle 1.7 suggests that a set of elements of $\mathbf{R}$ of infinite collective strength should be algebraically independent. On the other hand, a maximal such set generates $\mathbf{R}$ (Exercise 1.2). This suggests that $\mathbf{R}$ should abstractly be a polynomial ring. How can we prove this?

One special feature of polynomial rings is that they have many derivations (partial derivatives). With this in mind, we introduce the following idea:

Definition 4.1. A graded k-algebra R has *enough derivations* if for every non-zero homogeneous element $f \in R$ of positive degree there is a homogeneous derivation ∂ of negative degree such that $\partial(f) \neq 0$.

Example 4.2. Suppose that $R = k[y_i]_{i \in I}$ is a polynomial ring with k of characteristic 0. Then R has enough derivations: if f is a non-zero homogeneous element of positive degree then $\frac{\partial f}{\partial y_i}$ is non-zero for some i.

As the above example hints at, the concept of "enough derivations" isn't the right thing to use in positive characteristic. For the rest of the lecture, we therefore restrict to characteristic 0. (See the exercises for what happens in positive characteristic.)

Theorem 4.3. *Let R be a graded k-algebra. Then R is (isomorphic to) a polynomial ring if and only if it has enough derivations.*

Proof. It is enough to show that if R has enough derivations then it is a polynomial ring. We give a proof in the case that R is finitely generated; the general case uses the same ideas, but is technically a little more complicated (see Exercise 4.6).

Let $x_1, \ldots, x_n$ be homogeneous elements of R whose images form a k-basis of R_+/R_+^2. The x_i's generate R by Nakayama's lemma. Applying a permutation if necessary, we assume $\deg(x_1) \leq \cdots \leq \deg(x_n)$. We show that $x_1, \ldots, x_r$ are algebraically independent for $1 \leq r \leq n$ by induction on r. This will show that the map $k[T_1, \ldots, T_n] \to R$ given by $T_i \mapsto x_i$ is an isomorphism, where the T_i's are indeterminates.

Suppose that $x_1, \ldots, x_{r-1}$ are algebraically independent, but $x_1, \ldots, x_r$ are algebraically dependent. Thus there is a relation $0 = \sum_{i=0}^{d} a_i x_r^i$ with $a_i \in k[x_1, \ldots, x_{r-1}]$ such that $d > 0$ and $a_d \neq 0$. Of all such relations, choose a homogeneous one of minimal degree (by degree, we mean the common degree of each term). We will obtain a contradiction by producing a relation of smaller degree.

First suppose that a_d has positive degree. By assumption, there exists a derivation ∂ (homogeneous of negative degree) such that $\partial(a_d) \neq 0$. Applying ∂ to our relation yields a new relation of the form $0 = \partial(a_d) x_r^d + \cdots$ where the remaining terms have smaller degree in x_r. This contradicts the minimality of our initial relation. Thus a_d has degree 0, and we may as well assume $a_d = 1$.

Since the x_i are linearly independent modulo R_+^2, it follows that x_r does not belong to $k[x_1, \ldots, x_{r-1}]$. Thus $d > 1$ and $dx_r + a_{d-1}$ is non-zero. In particular, there is a derivation ∂ (homogeneous of negative degree) such that $\partial(dx_r + a_{d-1})$ is non-zero. Applying this to our original relation yields $0 = \partial(dx_r + a_{d-1})x_r^{d-1} + \cdots$ where the remaining terms have smaller degree in x_r. Note that since $\partial(x_r)$ has smaller degree than x_r, it belongs to $k[x_1, \ldots, x_{r-1}]$. This relation again contradicts the minimality of our original relation. We conclude that $x_1, \ldots, x_r$ are algebraically independent. □

COROLLARY 4.4. **R** *is (isomorphic to) a polynomial ring.*

Proof. It has enough derivations, just use $\frac{\partial}{\partial x_i}$ (defined on infinite series in the obvious manner). □

REMARK 4.5. Corollary 4.4 is true for any coefficient field k. This lecture only proves it when k has characteristic 0.

Exercises

Exercise 4.1. Show that the ultraproduct ring **S** has enough derivations, and is therefore a polynomial ring.

Exercise 4.2. Show that the equivalence "polynomial ring if and only if enough derivations" fails in both directions in positive characteristic. [Hint: show that $k[x]/(x^p)$ has enough derivations when k has characteristic p.]

Additional exercises

Exercise 4.3. Let R be a k-algebra. A *Hasse derivation* on R is a sequence $\{\partial_i\}_{i \ge 0}$ such that each ∂_i is k-linear and $\partial_n(xy) = \sum_{i+j=n} \partial_i(x)\partial_j(y)$. The intuition is that ∂_i should be like $\frac{1}{i!}\partial_1^i$. Construct a Hasse derivation on $k[x]$ (do not assume k has characteristic 0).

Exercise 4.4. Suppose k has characteristic $p > 0$ and is perfect (every element is a pth power). We say that R has *enough Hasse derivations* if whenever f is a homogeneous element that is not a pth power there is a homogeneous Hasse derivation ∂ of negative degree such that $\partial_1(f) \neq 0$. We have the following theorem [ESS2, Theorem 2.11]: a graded k-algebra is a polynomial ring if and only if it has enough Hasse derivations. The proof is a bit involved, so we'll just look at some special cases and adjacent results:

(a) Show that a polynomial ring has enough Hasse derivations.
(b) Show that if R has enough Hasse derivations then R is reduced. (This is the first step in the proof of the theorem.)
(c) Using the theorem, show that **R** is a polynomial ring.

Exercise 4.5. Let A_n be the exterior algebra on an n-dimensional vector space, regarded as a graded algebra, and let $\mathbf{A}$ be the inverse limit of the A_n's in the category of graded algebras.

(a) Show that $\mathbf{A}$ is *not* an exterior algebra on some vector space.
(b) Presumably, $\mathbf{A}$ should be a free algebra of a certain kind; can you figure out which kind?
(c) Prove your conjecture in (b). (As far as I know, this is still an open problem, and could make a nice little paper.)

Exercise 4.6. Prove Theorem 4.3 without the assumption that R is finitely generated. (The argument needs only minor modifications.)

Notes

Theorem 4.3 originally appeared in [ESS2, §2]. See [ESS3, §4.4] for a more expository treatement. As mentioned in the exercises, [ESS2] also proves a version of Theorem 4.3 for perfect fields of positive characteristic, though the proof is more involved. A later paper [ESS4] removes the perfectness hypothesis (and the argument is yet more complicated).

Chirvasitu and Hong [CH] have extended Corollary 4.4 to non-commutative rings and Lie algebras.