

Review of commutative algebra

In this lecture, we review some basic commutative algebra and also see how it extends to infinite variable polynomial rings. This is important to us since \mathbf{R} and \mathbf{S} are such rings.

Let R be a commutative ring. Recall that the (*Krull*) *dimension* of R is the supremum of n 's for which there is a strict chain of prime ideals $\mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_n$. We define the *dimension* of an ideal I , or its vanishing locus $V(I) \subset \text{Spec}(R)$, to be the Krull dimension of R/I .

EXAMPLE 5.1. The polynomial ring $R = k[x_1, \dots, x_n]$ has Krull dimension n . The chain

$$(0) \subset (x_1) \subset (x_1, x_2) \subset \cdots \subset (x_1, \dots, x_n)$$

shows that the dimension is at least n . The reverse inequality takes a little more effort to prove.

The dimension of I is defined in terms of chains of primes $I \subset \mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_n$. We can also define a notion of dimension by considering chains below I . First suppose that $I = \mathfrak{q}$ is itself a prime ideal. The *codimension* (or *height*) of \mathfrak{q} is the supremum of n 's for which there exists a strict chain $\mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_n = \mathfrak{q}$; equivalently, this is the Krull dimension of the localization $R_{\mathfrak{q}}$. For a general I , its *codimension* is the minimum codimension of a prime \mathfrak{q} containing I .

The above definitions behave well in “nice” rings. For example, suppose $R = k[x_1, \dots, x_n]$. Then the dimension of $V(I)$, as defined above, matches our intuition for the dimension of the algebraic set $V(I) \subset \mathbf{A}^n$. Furthermore, dimension and codimension sum to n in this case.

In general rings, (co)dimension isn't always well-behaved. However, suppose that R is a (possibly infinite) polynomial ring over a field k (or, more generally, a finitely generated k -algebra). Let I be a finitely generated ideal of R . The dimension of I will typically be infinite (Exercise 5.2), and so dimension is not a useful invariant. However, codimension is well-behaved in this setting. The basic reason for this is that I is extended from a finite variable subring (where one has all the familiar properties of codimension), and the extension process does not change codimension:

PROPOSITION 5.2. *Let A be a finitely generated k -algebra, let R be a polynomial ring over A , and let S be a polynomial ring over R . If I is a finitely generated ideal of R and J is its extension to S then $\text{codim}_R(I) = \text{codim}_S(J)$.*

COROLLARY 5.3. *Let R be as in the above proposition and let I be a finitely generated ideal of R . Then $\text{codim}_R(I)$ is finite.*

Proof. Let R_0 be the A -subalgebra of R generated by the variables appearing in a finite generating set of I . Then I is the extension of an ideal I_0 of R_0 , and so $\text{codim}_R(I) = \text{codim}_{R_0}(I_0)$ by the proposition. The right side is finite by the usual (finite variable) theory. \square

Here is an example of how one can use Proposition 5.2 to extend familiar results to the infinite setting.

PROPOSITION 5.4. *Let R be a polynomial ring over a finitely generated k -algebra A , and let $S = R[x]$. Let J be a finitely generated ideal of S that contains a monic polynomial in x , and let I be the contraction of J to R . Then I is finitely generated and $\text{codim}_R(I) = \text{codim}_S(J) - 1$.*

Proof. Choose a finite generating set for J , including the given monic polynomial f . Let R_0 be the A -subalgebra of R generated by all variables appearing in this generating set (other than x), and let $S_0 = R_0[x]$. Thus J is the extension of an ideal J_0 of S_0 . Let I_0 be the contraction of J_0 to R_0 . One easily sees that I is the extension of I_0 , and thus finitely generated. By classical (finite variable) theory, $\text{codim}_{R_0}(I_0) = \text{codim}_{S_0}(J_0) - 1$; the key point here is that $R_0 \rightarrow R_0[x]/(f)$ is finite and flat. Since $\text{codim}_{R_0}(I_0) = \text{codim}_R(I)$ and $\text{codim}_{S_0}(J_0) = \text{codim}_S(J)$ by Proposition 5.2, the result follows. \square

We now turn to regular sequences. Recall that elements f_1, \dots, f_r in a ring R form a *regular sequence* if f_i is a non-zero-divisor in the ring $R/(f_1, \dots, f_{i-1})$ for each $1 \leq i \leq r$. For example, x_1, \dots, x_n is a regular sequence in $R = k[x_1, \dots, x_n]$. The following proposition captures the most important property of regular sequences (though we will see other important properties later on):

PROPOSITION 5.5. *Let f_1, \dots, f_r be elements of $R = k[x_1, \dots, x_n]$. Then f_1, \dots, f_r form a regular sequence if and only if the ideal (f_1, \dots, f_r) has codimension r .*

Proof. If f_1, \dots, f_r form a regular sequence then Krull's principal ideal theorem implies that each equation $f_i = 0$ cuts down the codimension by one, and so (f_1, \dots, f_r) has codimension r . We leave the converse as an exercise. \square

Using Proposition 5.2, we can extend Proposition 5.5 to infinite polynomial rings:

PROPOSITION 5.6. *Let R be a (perhaps infinite) polynomial ring over k and let $f_1, \dots, f_r \in R$. Then f_1, \dots, f_r form a regular sequence if and only if (f_1, \dots, f_r) has codimension r .*

Proof. Let R_0 be a finite variable polynomial ring containing each f_i . Then f_1, \dots, f_r form a regular sequence in R_0 if and only if they form a regular sequence in R (obvious). Also, the codimension of the ideal generated by the f_i 's is the same in R_0 and R by Proposition 5.2. \square

Exercises

Exercise 5.1. Let $I = (xz, yz) \subset k[x, y, z]$. Determine the minimal prime ideals containing I and compute their codimensions.

Exercise 5.2. Let I be a finitely generated ideal in an infinite polynomial ring R . Show that I has infinite dimension. [This is intuitively obvious, but to actually prove it you have to exhibit long chains of prime ideals containing I .]

Exercise 5.3. Find three non-zero polynomials f_1, f_2, f_3 in $k[x, y, z]$ that are pairwise coprime but that do not form a regular sequence.

Additional exercises

Exercise 5.4. Let \mathfrak{p} be a prime of finite codimension c in a polynomial ring (over a field). Show that \mathfrak{p} is finitely generated. [Hint: to do the $c = 1$ case, contract down to a finite variable polynomial ring containing a non-zero element of \mathfrak{p} and then extend back. The general case can be proved by induction on c , using a similar idea.]

Exercise 5.5. Prove Proposition 5.2. First treat the case where I is prime. Then do the general case; you may need to make use of Exercise 5.4 in the general case.

Notes

See [ESS2, §3] for details about dimension theory in infinite polynomial rings.