Lecture 7 Overview of Rep Theory

We'll work over $\mathbb{C}$. $GL_n := GL_n(\mathbb{C})$. Special case: $GL_1 = \mathbb{C}^*$.

When you have an m-dim repn V of GL_n (i.e. $GL_n \circlearrowright V$), choosing a basis of V yields a homom. $GL_n \xrightarrow{\rho_V} GL_m$

Ex: $GL_n \circlearrowright \mathbb{C}^n$ std rep w/ std basis.

$GL_n \xrightarrow{\rho} GL_n$ is identity.

w/ another basis get $GL_n \xrightarrow{P(-)P^{-1}} GL_n$

Ex: $GL_2 \circlearrowright \mathbb{C}^2$ w/ basis $\{x_1, x_2\}$. So $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot x_1 = ax_1 + cx_2$.

Then $GL_2 \circlearrowright S^2(\mathbb{C}^2) = \{\text{homog. deg } 2 \text{ polys in } \mathbb{C}[x_1, x_2]\}$

basis $\{x_1^2, x_1x_2, x_2^2\}$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot x_1^2 = (ax_1 + cx_2)^2 = a^2x_1^2 + 2acx_1x_2 + c^2x_2^2$

So $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \xmapsto{\rho} \begin{array}{c} \begin{array}{ccc} x^2 & xy & y^2 \end{array} \\ \begin{bmatrix} a^2 & \cdot & \cdot \\ 2ac & \cdot & \cdot \\ c^2 & \cdot & \cdot \end{bmatrix} \end{array}$

polys in a, b, c, d

Ex: $GL_n \circlearrowright \mathbb{C}$ called the Determinant repn, Det.

$g \mapsto [\det g]$

Subex: $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto [ad - bc]$

Ex: $GL_n \circlearrowright \mathbb{C}$ denoted Det^5 (actually $\cong \underbrace{\mathrm{Det} \otimes \mathrm{Det} \otimes \ldots \otimes \mathrm{Det}}_{5} = \mathrm{Det}^{\otimes 5}$)

$g \mapsto [(\det g)^5]$

Def: V is called a poly repn of GL_n if entries of $\rho_V(g)$ are polys in entries of g.

~~rational repn~~ ~~rational functions~~

Note: indep. of choice of basis of V.

Ex: $GL_n \circlearrowright \mathbb{C}$ denoted Det^{-5}, rational but not poly.

$g \mapsto [(\det g)^{-5}]$

Note: $\mathrm{Det}^{-1} \cong (\mathrm{Det})^*$.

Ex: $GL_n \circlearrowright (\mathbb{C}^n)^*$ is rational, not poly (see supp. exercise)

Easy observation: V, W both poly (resp. rational) then so is $V \oplus W$, $V \otimes W$, or any submod / quotient / direct summand of V.

Ex: $GL_2\mathbb{C} \circlearrowright \mathbb{C}^2 \otimes \mathbb{C}^2$ $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto$

$$\begin{array}{ccccc} & x_1\otimes x_1 & x_1\otimes x_2 & x_2\otimes x_1 & x_2\otimes x_2 \\ & \left[\begin{matrix} a^2 \\ ac \\ ac \\ c^2 \end{matrix}\right. & \begin{matrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{matrix} & \begin{matrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{matrix} & \left.\begin{matrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{matrix}\right] \end{array}$$

remember, $g(v \otimes w) = g(v) \otimes g(w)$

$x_1 \otimes x_1 \mapsto (ax_1 + cx_2) \otimes (ax_1 + cx_2)$

(Note: All entries in $(\mathbb{C}^n)^{\otimes k}$ are polys homog of degree k, same for summands.

Ex: $\mathrm{Det} \cong \Lambda^n(\mathbb{C}^n) \overset{\oplus}{\subset} (\mathbb{C}^n)^{\otimes n}$, det is poly of degree n.)

So what isn't rational? Nice segue to next topic: $GL_1 = \mathbb{C}^*$.

As an abelian group, $\mathbb{C}^*$ is wacky! "Easy" to make awful gp homom $\mathbb{C}^* \to \mathbb{C}^*$.

(But not continuous! Smooth! Rational reps are ~~[illegible]~~ nice!)

Ex: $\mathbb{C}^* \to \mathbb{C}^*$ is NOT rational. (Rational is holomorphic.)

$z \mapsto \bar{z}$ or $|z|$

Rational ends up the same as smooth $\mathbb{C}$x (i.e. holomorphic), rules out nonsense.

What 1D rat'l reps are there?

$[z] \mapsto \left[\frac{p(z)}{q(z)}\right]$ but only roots of q is zero (or not defined on $\mathbb{C}^*$)

and ~~=~~ p ~~=~~ (or not invertible)

$\Rightarrow$ both monomials!

So $[z] \mapsto [z^k]$ for $k \in \mathbb{Z}$!! (Actually Det^k.)

(You might have expected continuous behavior, but actually discrete!)

$$\begin{array}{ccc} \text{Rat'l 1D reps of } GL_1 \text{ (up to isom)} & \longleftrightarrow & \mathbb{Z} \\ \cup & & \cup \\ \text{Poly 1D} & \longleftrightarrow & \mathbb{Z}_{\geq 0} \end{array}$$

⌜Rmk: Just as q was z^l above, for rat'l rep of GL_n, q (denominators) must be a power of det in order to not vanish for any invertible matrix!⌟

$\rightsquigarrow$ any rat'l rep is $V \otimes \mathrm{Det}^{-l}$ for V poly rep!!

Now consider $GL_1 \curvearrowright V$.

Key idea: Commuting matrices have simult. evector!

$G \curvearrowright V$, 1D subreps $\Leftrightarrow$ (span of) simult. evectors for $\forall g \in G$.

Since $p(z_1), p(z_2)$ commute for $z_1, z_2 \in \mathbb{C}^*$, V has 1D subrep.

so all irreps (rational) are 1D!

Less obvious: can't define any rational Jordan blocks!

Aside: $z \mapsto \begin{bmatrix} 1 & z \\ 0 & 1 \end{bmatrix}$ is homom from $(\mathbb{C}, +)$ not $(\mathbb{C}^*, \times)$.

Thm: $GL_1 \curvearrowright V$ then $V = \bigoplus$ 1D subreps (i.e. V is semisimple)

Awesome, harder thm:

Thm: $GL_n \curvearrowright V$ (rational) then $V = \bigoplus$ irreps.

Irreps of GL_1 had nice parametrization... want GL_n too.

First look at $T_n \subset GL_n$, $T_n = \{\text{diagonal matrices}\} = \left\{\begin{bmatrix} z_1 & & 0 \\ & \ddots & \\ 0 & & z_n \end{bmatrix}\right\} \cong (\mathbb{C}^\times)^n$

$$\text{1D rat'l reps of } T_n \Longleftrightarrow \mathbb{Z}^n \ni \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$$

$$\underbrace{\begin{bmatrix} z_1 & & 0 \\ & \ddots & \\ 0 & & z_n \end{bmatrix}}_{= t} \longmapsto [z_1^{\lambda_1} z_2^{\lambda_2} \cdots z_n^{\lambda_n}]$$

$\chi_\lambda(t) \in \mathbb{C}^\times$.

= Irreps b/c $\exists$ simult. evector.

called a *weight*

It records simult. evalue.

<u>Note:</u> Poly rep $\Leftrightarrow$ $\lambda_i \geq 0$ $\forall i$. We call λ a <u>polynomial weight</u>.

Any rat'l rep of GL_n restricts to rat'l rep of T_n so splits into ~~[illegible]~~ simult. eigenspaces

$$V = \bigoplus_{\lambda \in \mathbb{Z}^n} V_\lambda \qquad V_\lambda = \{ v \in V \mid t \cdot v = \chi_\lambda(t) v \quad \forall t \in T_n \}$$

as v.s. and T_n-rep NOT GL_n-rep

weight vector

weight space

<u>Ex:</u> $GL_n \circlearrowright \mathbb{C}^n = \mathbb{C}\langle x_1 \rangle \oplus \cdots \oplus \mathbb{C}\langle x_n \rangle$

x_i

$$\begin{bmatrix} z_1 & & \\ & \ddots & \\ & & z_n \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = z_1 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

evalue $z_1^1 z_2^0 \cdots z_n^0 = \chi_\lambda(t)$ for $\lambda = (1, 0, 0, \ldots, 0)$

x_i has weight $\varepsilon_i = (0, \ldots, 1, \ldots, 0)$ ← i^{th} spot.

$$\mathrm{wts}(\mathbb{C}^n) = \{\varepsilon_i\}_{i=1}^n$$

Ex: $GL_2 \circlearrowright \mathbb{C}^2\otimes\mathbb{C}^2$

$$\begin{bmatrix} z_1 & \\ & z_2 \end{bmatrix}\cdot(x_1\otimes x_2) = z_1x_1\otimes z_2x_2 = z_1z_2(x_1\otimes x_2) \qquad \text{weight } (1,1)=\varepsilon_1+\varepsilon_2$$

$$wt(v\otimes w) = wt(v)+wt(w)$$

Also $wt(x_2\otimes x_1) = (1,1)$ $\qquad$ $\dim V_{(1,1)} = 2$ multiplicity.

$$wts(\mathbb{C}^2\otimes\mathbb{C}^2) = \{(2,0), (1,1), (1,1), (0,2)\} \quad \text{multiset.}$$

Idea: Multisets add in $\oplus$.

Ex: $\mathbb{C}^2\otimes\mathbb{C}^2 \cong S^2(\mathbb{C}^2)\oplus\Lambda^2(\mathbb{C}^2)$

weight basis $\begin{Bmatrix} x_1^2 \\ x_1x_2 \\ x_2^2 \end{Bmatrix}$ $\{x_1\wedge x_2\}$ $\quad$ weights $\begin{Bmatrix} (2,0) \\ (1,1) \\ (0,2) \end{Bmatrix}$ $\{(1,1)\}$

Idea: $S_n \subset GL_n$ conjugation preserves T_n and permutes entries.

$\rightsquigarrow$ if $v\in V_\lambda$, $w\in S_n$ then $w\cdot v\in V_{w\cdot\lambda}$ $\rightsquigarrow$ $wts(V)$ is S_n-invt.

Classification

Def: $\lambda\in\mathbb{Z}^n$ is dominant if $\lambda_1\geq\lambda_2\geq\ldots\geq\lambda_n$ $\quad$ (≥ 0 iff poly too)

Ex: $(2,0)$ and $(1,1)$ but not $(0,2)$

Def: $\lambda\geq\mu$ (partial order) if $\lambda-\mu = \sum_{i=1}^{n-1} c_i(\varepsilon_i-\varepsilon_{i+1})$ for $c_i\geq 0$.

Ex: $(2,0) > (1,1) > (0,2)$ $\quad$ since $(2,0)-(1,1) = (1,-1) = \varepsilon_1-\varepsilon_2$

(moving 1 to left is adding $\varepsilon_i-\varepsilon_{i+1}$.)

Ex: $(2,0) \not\geq (1,0)$

Ex: $(1,0,0,1)$ and $(0,1,1,0)$ incomparable in $\mathbb{Z}^4$.

Thm: Irred Ratl GL_n reps up to isom $\Longleftrightarrow$ Dominant Weights

$$L_\lambda \longleftrightarrow \lambda$$

$$V \longmapsto hw(V)$$

- λ is a hw if $\lambda \geq \mu$ $\forall \mu \in wts(V)$.
  ($\lambda \in wts(V)$ and)

Ex: $hw(S^2(\mathbb{C}^2)) = (2,0)$
$hw(\Lambda^2(\mathbb{C}^2)) = (1,1)$

Thm says irrep V has ! hw, and hw determines irrep up to isom, and hw is dominant, and any dominant wt is irrep...

Moreover:
- $\dim(V_\lambda) = 1$ for hw.
- λ poly $\Leftrightarrow$ V poly.
- $\sum \lambda_i = m \Leftrightarrow V \overset{\oplus}{\subset} (\mathbb{C}^n)^{\otimes m}$.

Ex: Irreps of $GL_2 \longleftrightarrow \mathbb{Z} \times \mathbb{Z}$

$S^m(\mathbb{C}^2) \longleftrightarrow (m,0)$

basis: $x_1^m \quad x_1^{m-1}x_2 \quad \cdots \quad x_2^m$

wts: $(m,0) \quad (m-1,1) \quad \cdots \quad (0,m)$

$\mathrm{Det} \longleftrightarrow (1,1)$

$\mathrm{Det}^{-5} \longleftrightarrow (-5,-5)$

$(6,3) = (3,0) + (3,3)$

$L_{(6,3)} \cong L_{(3,0)} \otimes \mathrm{Det}^3$.

Ex: $GL_2 \circlearrowright \underbrace{(\mathbb{C}^2)^{\otimes 3}}_{8\text{-dim}} = S^3(\mathbb{C}^2) \oplus ?$

wts:	$(3,0)$	$(2,1)$	$(1,2)$	$(0,3)$
mult:	1	3	3	1
mult in S^3:	1	1	1	1

$wts(?) =$

$(2,1)$	$(1,2)$
2	2

hw in summand of ?, must be (2,1) since (1,2) not dominant!

$$L_{(2,1)} = L_{(1,0)} \otimes \mathrm{Det} = \mathbb{C}^2 \otimes \mathrm{Det}$$

wts	(2,1)	(1,2)
mult	1	1

so $? = (\mathbb{C}^2 \otimes \mathrm{Det}) \oplus (\mathbb{C}^2 \otimes \mathrm{Det})$

$$\Rightarrow \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \cong S^3(\mathbb{C}^2) \oplus (\mathbb{C}^2 \otimes \mathrm{Det}) \oplus (\mathbb{C}^2 \otimes \mathrm{Det})$$