LECTURE 7

Representations of $\mathbf{GL}_n$

For the next two lectures, we work over the complex numbers $\mathbf{C}$. Let V be a finite dimensional complex representation of the group $\mathbf{GL}_n$ of $n \times n$ invertible complex matrices. Picking a basis $v_1, \dots, v_m$ for V, the action of $\mathbf{GL}_n$ corresponds to a group homomorphism $\rho\colon \mathbf{GL}_n \to \mathbf{GL}_m$. For $g \in \mathbf{GL}_n$, let $\rho_{i,j}(g)$ be the (i,j) entry of the matrix $\rho(g)$. We say that the representation V is *rational* (resp. *polynomial*) if $\rho_{i,j}(g)$ is a rational (resp. polynomial) function of the entries of g. Of course, any polynomial representation is rational.

Here are some simple observations:

- The notion of rational or polynomial is independent of the choice of basis.
- The classes of rational and polynomial representations are closed under direct sums, tensor products, and passing to sub or quotients.
- The *standard representation* of $\mathbf{GL}_n$ is the vector space $\mathbf{C}^n$ equipped with its usual $\mathbf{GL}_n$-action. It is a polynomial representation.
- For each $k \in \mathbf{Z}$ there is a 1-dimensional representation of $\mathbf{GL}_n$ corresponding to the homomorphism $\mathbf{GL}_n \to \mathbf{GL}_1$ given by $g \mapsto (\det g)^k$. This is rational for all k and polynomial for $k \geq 0$.

The following is an extremely important fact about rational representations (see Exercise 7.8 for the proof).

THEOREM 7.1. *Every rational representation of $\mathbf{GL}_n$ decomposes into a direct sum of irreducible representations; i.e., rational representations are* semisimple.

Thus to understand the basic structure of rational representations, it is enough to understand the irreducible ones. For $\mathbf{GL}_1$ this is straightforward. For $k \in \mathbf{Z}$, let $\chi_k\colon \mathbf{GL}_1 \to \mathbf{GL}_1$ be the homomorphism given by $\chi_k(t) = t^k$. Then the χ_k are exactly the irreducible rational representations of $\mathbf{GL}_1$ (Exercise 7.5).

There is a mild generalization of the $\mathbf{GL}_1$ story that is very useful. Let $\mathbf{T}_n = (\mathbf{GL}_1)^n$; a group of this form is called an *algebraic torus*. A *weight* of $\mathbf{T}_n$ is an n-tuple of integers. We let $X(\mathbf{T}_n) = \mathbf{Z}^n$ be the set of all weights (the *weight lattice*). For $\lambda \in X(\mathbf{T}_n)$, let $\chi_\lambda\colon \mathbf{T}_n \to \mathbf{GL}_1$ be the homomorphism given by $\chi_\lambda(t) = t_1^{\lambda_1} \cdots t_n^{\lambda_n}$. Then once again, every rational representation of $\mathbf{T}_n$ decomposes into irreducibles, and the χ_λ account for all rational irreducible representations. (The definition of rational representation here is analogous to the $\mathbf{GL}_n$ case.)

The above result is often reformulated in the following manner. Let V be a rational representation of $\mathbf{T}_n$. Let V_λ be the set of all vectors $v \in V$ such that $t \cdot v = \chi_\lambda(t) v$ for all $t \in \mathbf{T}_n$. This is called the λ-*weight space* of V. Then V decomposes as a direct sum $\bigoplus_{\lambda \in X(\mathbf{T}_n)} V_\lambda$. We say that λ is a weight of V if $V_\lambda \neq 0$, and define its multiplicity to be $\dim(V_\lambda)$.

Now let V be a rational representation of $\mathbf{GL}_n$. Thinking of $\mathbf{T}_n$ as the group of diagonal matrices in $\mathbf{GL}_n$, we can restrict V to $\mathbf{T}_n$ and look at its weights. While $\mathbf{T}_n$ is much smaller than $\mathbf{GL}_n$, the weights of V nonetheless tell us a lot about V. To make this precise, we introduce some terminology:

- We say that a weight λ is *dominant* if $\lambda_1 \geq \cdots \geq \lambda_n$.
- We say that a weight λ is *higher* than a weight μ, written $\lambda \geq \mu$, if $\lambda - \mu$ is a $\mathbf{Z}$-linear combination of the vectors $e_i - e_j$ for $i < j$, where e_i is the ith standard basis vector of $\mathbf{Z}^n$.

We can now state a second important theorem. The proof of this theorem is a bit involved, and we simply refer to [FH, §14,15].

Theorem 7.2. *Let V be an irreducible rational representation of $\mathbf{GL}_n$.*

(a) There exists a unique weight λ of V that is higher than all other weights; it is called the highest weight *of V.*
(b) The highest weight space V_λ is one-dimensional.
(c) The highest weight determines V up to isomorphism: if W is a second irreducible rational representation with highest weight μ then $V \cong W$ if and only if $\lambda = \mu$.
(d) The representation V is polynomial if and only if $\lambda_i \geq 0$ for all i.

Furthermore, the weights that occur as the highest weight of some irreducible rational representation are exactly the dominant weights.

Exercises

Exercise 7.1. Compute the weights of $\mathrm{Sym}^2(\mathbf{C}^3)$ as a representation of $\mathbf{GL}_3$, and draw the Hasse diagram of the dominance order.

Exercise 7.2. Compute the multiset of weights of $(\mathbf{C}^2)^{\otimes 4}$ as a representation of $\mathbf{GL}_2$. Decompose this representation into irreducible representations.

Exercise 7.3. This exercise walks you through the decomposition of the 27-dimensional representation $V = (\mathbf{C}^3)^{\otimes 3}$ of $\mathbf{GL}_3$.

(a) Compute the multiset of weights of $\bigwedge^2(\mathbf{C}^3)$, and then the multiset of weights of $W = \bigwedge^2(\mathbf{C}^3) \otimes \mathbf{C}^3$. Deduce that $W = L_{(2,1,0)} \oplus \bigwedge^3(\mathbf{C}^3)^{\oplus m}$ for some unknown multiplicity m. If you know m, how do you compute the multiset of weights of $L_{(2,1,0)}$?
(b) With what multiplicity does $\bigwedge^3(\mathbf{C}^3)$ appear as a direct summand of V? [Hint: This one-dimensional representation can only appear as a summand of a particular six-dimensional weight space. The symmetric group $S_3 \subset \mathbf{GL}_3$ acts in a particular way on this six-dimensional space, and on $\bigwedge^3(\mathbf{C}^3)$.]
(c) Use part (b) to compute m and the weights of $L_{(2,1,0)}$.
(d) Now compute the decomposition of $\mathrm{Sym}^2(\mathbf{C}^3) \otimes \mathbf{C}^3$ into irreducible representations.

(e) Now compute the decomposition of V into irreducible representations. [Hint: First decompose $\mathbf{C}^3 \otimes \mathbf{C}^3$, then tensor with $\mathbf{C}^3$.]

(f) (Come back after Lecture 8) Match your answer with the answer that Schur–Weyl duality gives.

Additional exercises

Exercise 7.4. If G acts on V, then G acts on $V^* = \{f \colon V \to \mathbf{C}\}$ by the formula $g \cdot f(v) = f(g^{-1}v)$.

(a) For $G = \mathbf{GL}_2$ and $V = \mathbf{C}^2$, write down the matrix for the action of $g \in G$ on V^*, with respect to the dual standard basis.

(b) From this you might be able to guess the general formula which takes the matrix of g on a basis of V and produces the matrix of g on the dual basis of V^*. Prove that $(\mathbf{C}^n)^* \otimes \det$ is a polynomial representation of $\mathbf{GL}_n$, whereas $(\mathbf{C}^n)^*$ is only rational (for $n > 1$).

Exercise 7.5. Prove the classification of irreducible rational representations of $\mathbf{GL}_1$.

Exercise 7.6. Let $\rho \colon \mathbf{GL}_1 \to \mathbf{GL}_2$ be a homomorphism given by

$$\rho(z) = \begin{pmatrix} z^k & r(z) \\ 0 & z^k \end{pmatrix}$$

for some $k \in \mathbf{Z}$ and some rational function $r(z)$. Deduce that $r = 0$. This does not prove the semisimplicity theorem for $\mathbf{GL}_1$, but it certainly makes it believable. [Hint: Compare certain coefficients in $\rho(z)^2 = \rho(z^2)$.]

Exercise 7.7. Let $Z \subset \mathbf{GL}_n$ be the group of scalar matrices; this is isomorphic to $\mathbf{GL}_1$. Let V be a rational representation of $\mathbf{GL}_n$ and let $V = \bigoplus_{k \in \mathbf{Z}} V_k$ be the weight decomposition for Z. Show that each V_k is a $\mathbf{GL}$-subrepresentation of V. We thus see that every rational $\mathbf{GL}$-representation admits a canonical $\mathbf{Z}$-grading.

Exercise 7.8. Let V be a rational representation of $\mathbf{GL}_n$.

(a) Let $\langle , \rangle_0$ be a Hermitian form on V. Define a new Hermitian form $\langle , \rangle$ on V by the formula

$$\langle v, w \rangle = \int_{\mathbf{U}_n} \langle gv, gw \rangle_0 dg.$$

Here $\mathbf{U}_n \subset \mathbf{GL}_n$ is the unitary group and dg is its Haar measure. Show that $\langle , \rangle$ is $\mathbf{GL}_n$-invariant, i.e., $\langle gv, gw \rangle = \langle v, w \rangle$ for $g \in \mathbf{GL}_n$ and $v, w \in V$. [Hint: $\mathbf{U}_n$ is Zariski dense in $\mathbf{GL}_n$.]

(b) Let W be a $\mathbf{GL}_n$-subrepresentation of V and let W' be its orthogonal complement under $\langle , \rangle$. Show that W' is also a $\mathbf{GL}_n$-subrepresentation and that $V = W \oplus W'$.

(c) Show that V decomposes as a direct sum of irreducible representations (Theorem 7.1).