Lecture 8, Schur-Weyl Duality + Schur Functors

①

Motivation: $S^k(\mathbb{C}^n) \overset{\oplus}{\subset} (\mathbb{C}^n)^{\otimes k} \overset{\oplus}{\supset} \Lambda^k(\mathbb{C}^n)$ two useful $\overset{GL_n}{\wedge}$summands, and

can construct them not with GL_n theory but with S_k.

The sym. gp. $S_k \curvearrowright (\mathbb{C}^n)^{\otimes k}$ by permuting tensor factors. $(12)\cdot(x_1\otimes x_3\otimes x_2) = x_3\otimes x_1\otimes x_2$

$S^k(\mathbb{C}^n) = \mathrm{Sym}^k(\mathbb{C}^n) = \{ v\in(\mathbb{C}^n)^{\otimes k} \mid wv=v \ \forall w\in S_k\}$ the S_k-<u>invariants</u>

$\Lambda^k(\mathbb{C}^n) = \{ v\in(\mathbb{C}^n)^{\otimes k} \mid wv=\mathrm{sgn}(w)v \ \forall w\in S_k\}$ the S_k-<u>antiinvariants</u>

all copies of triv inside $(\mathbb{C}^n)^{\otimes k}$,

// sgn //

What about other reps of S_k, do they help too? Sure do.

(If your experience with Rep S_k has been via character tables, you're probably afraid)(Unnecessarily.)

Quick primer on Rep S_k:

(same size, but not usually bijection)

$\mathrm{Irr}_{\mathbb{C}} S_k \overset{?}{\cancel{\leftrightarrow}}$ Conj classes $\leftrightarrow$ partitions of n

$\qquad\qquad\qquad\qquad\quad \parallel$

cycle types.

Def: A <u>partition of k</u> is $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq 0)$ with $\sum \lambda_i = k$. Write $\lambda \vdash k$.

<u>Ex:</u> some cycle types in S_5

$(123)(45)$ $\quad 3+2 \quad (3,2,0,0,\dots)$

$(12)(3)(4)(5)$ $\quad 2+1+1+1 \quad (2,1,1,1,0,\dots)$

$\leftarrow$ at most k can be nonzero.

Reln to GL_n wts:

- # of rows = # of nonzero λ_i ($\leq k$ if $\lambda\vdash k$)
- Partitions of <u>any size</u> with $\leq n$ rows $\longleftrightarrow$ dominant poly wts for GL_n

It turns out you can explicitly construct a rep. associated to $\lambda \vdash n$, Specht 19__.

tableau $\begin{array}{|c|c|c|}\hline 1&5&3\\ \hline 4&2\\ \cline{1-2}\end{array} = T$ (any labeling $1\dots k$) $\rightsquigarrow$ polynomial in $\mathbb{C}[x_1\dots x_k]$

$P_T = (x_1-x_4)(x_5-x_2)$

$\begin{array}{|c|c|}\hline 3&1\\ \hline 2&4\\ \hline 5\\ \cline{1-1}\end{array} = T' \rightsquigarrow P_{T'} = (x_3-x_2)(x_3-x_5)(x_2-x_5)(x_1-x_4)$

S_k acts on tableau and polys by permuting labels.

Def: $\mathcal{S}_\lambda := \operatorname{Span}\{p_T \mid \operatorname{shape}(T)=\lambda\}$

Ex: $\lambda = \begin{array}{|c|c|c|c|}\hline \ & \ & \ & \ \\ \hline \ \\ \cline{1-1}\end{array}$ if $T = \begin{array}{|c|c|}\hline i & \text{rest} \\ \hline j \\ \cline{1-1}\end{array}$ then $p_T = x_i - x_j$

$\mathcal{S}_\lambda = \operatorname{Span}\{x_i - x_j\}$ inside polys of degree 1. Standard repn

(thinking of all polys of deg 1 as $\mathbb{C}^k$, this is kernel of $\mathbb{C}^k \to \text{triv}$)
$x_i \mapsto 1 \ \forall i.$

Now $\mathcal{S}_{\begin{array}{|c|c|c|c|}\hline \ & \ & \ & \ \\ \hline \ \\ \cline{1-1}\end{array}}$ has a basis $\{x_1 - x_i\}_{i \geq 2} = \left\{ p_{\begin{array}{|c|c|c|c|}\hline 1 & 3 & 4 & 5 \\ \hline 2 \\ \cline{1-1}\end{array}},\ p_{\begin{array}{|c|c|c|c|}\hline 1 & 2 & 4 & 5 \\ \hline 3 \\ \cline{1-1}\end{array}},\ p_{\begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 5 \\ \hline 4 \\ \cline{1-1}\end{array}},\ p_{\begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 5 \\ \cline{1-1}\end{array}} \right\}$

Thm: 1) $\mathcal{S}_\lambda$ has a basis $\{p_T \mid \operatorname{shape}(T) = \lambda,\ T \text{ is standard Young tableau}\}$.

meaning $\text{increasing}\downarrow \overset{\text{increasing}}{\begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 \\ \cline{1-2}\end{array}}$. ⇝ formulas for $d_\lambda = \dim \mathcal{S}_\lambda$

2) $\mathcal{S}_\lambda$ is irreducible, and $\operatorname{Irr} S_k \leftrightarrow \{\mathcal{S}_\lambda\}_{\lambda \vdash k}$.

There are many ~~[illegible]~~ nice formulas and other nice bases (Young seminormal form...)

Ref: Okounkov-Vershik, New approach ... II.

⇝ even formulas for idempotents in $\mathbb{C}[S_k]$ which project to the $\mathcal{S}_\lambda$-part of $\mathbb{C}[S_k] \cong \prod_{\lambda \vdash k} \operatorname{Mat}_{d_\lambda \times d_\lambda}(\mathbb{C})$

Ref: F-H.

Don't be afraid...

Let's look again at $(\mathbb{C}^2)^{\otimes 3}$ 8-dim'l.

$$GL_2 \circlearrowright (\mathbb{C}^2)^{\otimes 3} \circlearrowleft S_3$$

g acts by $g\otimes g\otimes g$ w acts to shuffle $\otimes$ factors these actions commute!

As repn of GL_2, $(\mathbb{C}^2)^{\otimes 3} \cong \underset{\dim 4}{L_{(3,0)}} \oplus \underset{\dim 2}{L_{(2,1)}^{\oplus 2}}$

As repn of S_3, $(\mathbb{C}^2)^{\otimes 3} \cong \mathbb{S}_{\square\square\square}^{\oplus 4} \oplus \mathbb{S}_{\begin{smallmatrix}\square&\square\\ \square\end{smallmatrix}}^{\oplus 2}$.

$(3,0) \rightarrow$ triv dim=1 standard dim=2 $\leftarrow (2,1)$

Multiplicity of $\mathbb{S}_{\begin{smallmatrix}\square\\ \square\\ \square\end{smallmatrix}}$ is zero

$\mathbb{S}_{\begin{smallmatrix}\square\\ \square\\ \square\end{smallmatrix}}$ is sgn rep,

antiinvariants are $\Lambda^3(\mathbb{C}^2) = 0$.

<u>Thm</u> (Schur-Weyl duality):

As (GL_n, S_k)-bimodules, $$(\mathbb{C}^n)^{\otimes k} = \bigoplus_{\substack{\lambda \vdash k \\ \text{w/ } \leq n \\ \text{rows}}} (L_\lambda \boxtimes \mathbb{S}_\lambda)$$

a subset of Irr S_k $\nearrow$ $\nwarrow$ ~~[illegible]~~ a subset of dominant poly wts of GL_n

$\boxtimes$ just means $\underset{\mathbb{C}}{\otimes}$ but <u>not</u> as repns

$$\underset{GL_n \circlearrowright}{L_\lambda} \boxtimes \underset{\circlearrowleft S_k}{\mathbb{S}_\lambda}$$

so $g\cdot(v\boxtimes m) = gv\boxtimes m$ $g\in GL_n$

$w\cdot(v\boxtimes m) = v\boxtimes wm$ $w\in S_k$

From GL_n perspective, $\mathbb{S}_\lambda$ is just a vector space of dim d_λ,

$L_\lambda \boxtimes \mathbb{C}^3 = L_\lambda \oplus L_\lambda \oplus L_\lambda$ records multiplicity

From S_k perspective L_λ is just v/s recording multiplicity

$$(\mathbb{C}^2)^{\otimes 3} = (L_{(3,0)} \boxtimes \underset{\uparrow \dim 1}{\mathbb{S}_{(3,0)}}) \oplus (L_{(2,1)} \boxtimes \underset{\uparrow \dim 2}{\mathbb{S}_{(2,1)}})$$

and vice versa.

so $L_{(3,0)}$ once so $L_{(2,1)}$ twice

Another way to see things: Schur-think.

Schur's lemma: $\mathrm{Hom}(L_\lambda, L_\mu) = \begin{cases} \mathbb{C}\text{-id} & \lambda = \mu \\ 0 & \text{else} \end{cases}$.

Def: Fix any decomp $V = \bigoplus L_\lambda^{\oplus m_\lambda}$. The λ-isotypic component is the submodule $L_\lambda^{\oplus m_\lambda} =: I_\lambda \circlearrowleft GL_n$ (Ex: $\mathrm{Sym}^k(\mathbb{C}^n) = \mathbb{S}_{\square\square\square\square\square}$-isotypic component of $(\mathbb{C}^n)^{\otimes k}$)

Claim: I_λ is indep of choice of direct sum decomp (whereas any given summand wouldn't be, e.g. one line in $\mathrm{Sym}^k(\mathbb{C}^n)$

Pf/Reason: Because $I_\lambda = \mathrm{Span}\{\mathrm{Image}(L_\lambda \to V) \text{ over all maps}\}$.

By Schur, no maps to other summands; $\mathrm{Hom}(L_\lambda, V)$ is spanned by inclusions into each of m_λ factors. $\dim \mathrm{Hom}(L_\lambda, V) = m_\lambda$.

In fact, I_λ is canonically $\underset{GL_n}{\overset{}{I_\lambda}} \cong \underset{GL_n}{L_\lambda} \boxtimes \mathrm{Hom}(L_\lambda, V)$! $\boxed{\varphi(v) \leftarrow\!\shortmid v \otimes \varphi}$

plain v.s. of dim m_λ.

Def: $\mathrm{Hom}(L_\lambda, V)$ is the multiplicity space of L_λ in V.

Thm: $V \underset{\text{canonically}}{=} \bigoplus I_\lambda = \bigoplus L_\lambda \boxtimes \mathrm{Hom}(L_\lambda, V)$

Now if H is a group / A is an algebra $\circlearrowright V$ and commuting with GL_n

then $A \circlearrowright \mathrm{Hom}(L_\lambda, V)$ via $L_\lambda \xrightarrow{\varphi} V \xrightarrow{a\cdot} V$

$a \cdot \varphi = a \circ \varphi$

mult spaces become A-reps. $V = \bigoplus \underset{GL_n}{L_\lambda} \boxtimes \underset{A}{\mathrm{Hom}(L_\lambda, V)}$ as (GL_n, A)-bimodules

Final thoughts: If $A \twoheadrightarrow \mathrm{End}_{GL_n}(V)$ then image is semisimple and mult. spaces are irred over A. More Schur-think.

So Schur-Weyl duality is really a consequence of the easier theorem

Thm: $\mathbb{C}[S_k] \twoheadrightarrow \mathrm{End}_{GL_n}\left((\mathbb{C}^n)^{\otimes k}\right)$

Conclusion: Inside $(\mathbb{C}^n)^{\otimes k} = \bigoplus L_\lambda \boxtimes \mathcal{S}_\lambda$ we have another construction of L_λ and $\mathcal{S}_\lambda$, namely

$$\mathcal{S}_\lambda = \mathrm{Hom}_{GL_n}\left(L_\lambda, (\mathbb{C}^n)^{\otimes k}\right) \circlearrowleft S_k$$

$$L_\lambda = \mathrm{Hom}_{S_k}\left(\mathcal{S}_\lambda, (\mathbb{C}^n)^{\otimes k}\right) \circlearrowleft GL_n$$

(generalizes $L_{\square\square\square\square} = \mathrm{Sym}^k(\mathbb{C}$
$L_{\begin{smallmatrix}\square\\\square\\\square\\\square\end{smallmatrix}} = \Lambda^k(\mathbb{C}^n)$)

Def: For any v.s. V and $\lambda \vdash k$ define $\mathbb{S}_\lambda(V) := \mathrm{Hom}_{S_k}(\mathcal{S}_\lambda, V^{\otimes k})$

If $\dim V = n$ so $GL(V) = GL_n$, then $\mathbb{S}_\lambda(V) = \begin{cases} L_\lambda & \text{if } \lambda \text{ has } \leq n \text{ rows} \\ 0 & \text{else.} \end{cases}$

(like $\Lambda^3(\mathbb{C}^2)$
$\qquad \| $
$\mathbb{S}_{\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}}(\mathbb{C}^2)$

But $\mathbb{S}_\lambda(-)$ is functorial, a functor $\mathrm{Vect} \to \mathrm{Vect}$. Called a <u>Schur functor</u>.

$$\begin{tikzcd} \mathrm{Vect} \arrow[r] & \mathrm{Vect}. \\ V \arrow[u, phantom, "\in"] \arrow[r, mapsto] & \mathbb{S}_\lambda(V) \arrow[u, phantom, "\in"] \\ GL(V) \arrow[u, phantom, "\circlearrowright"] \arrow[r, squiggly] & GL(V) \arrow[u, phantom, "\circlearrowright"] \end{tikzcd}$$

via $\mathbb{S}_\lambda(g)$

(or $\mathrm{End}(V)$) ($\mathrm{End}(V)$)

The fact that $GL(V) \circlearrowright \mathbb{S}_\lambda(V)$ follows from this!

S-W duality rephrased again: $T^k(-) \cong \bigoplus_{\lambda \vdash k} \mathbb{S}_\lambda(-) \boxtimes \mathcal{S}_\lambda$
$(T^k(V) = V^{\otimes k})$

Def: Polynomial functors $= \bigoplus$ Schur functors $=$ $\oplus$ of $\oplus$ of T^k.