LECTURE 8

Polynomial functors

Two important examples of polynomial representations of $\mathbf{GL}_n$ are the symmetric powers $\mathrm{Sym}^k(\mathbf{C}^n)$ and exterior powers $\bigwedge^k(\mathbf{C}^n)$. We have isomorphisms

$$\mathrm{Sym}^k(\mathbf{C}^n) \cong \mathrm{Hom}_{\mathfrak{S}_k}(\mathrm{triv}, (\mathbf{C}^n)^{\otimes k}), \qquad \textstyle\bigwedge^k(\mathbf{C}^n) \cong \mathrm{Hom}_{\mathfrak{S}_k}(\mathrm{sgn}, (\mathbf{C}^n)^{\otimes k}),$$

where $\mathfrak{S}_k$ denotes the symmetric group, and triv and sgn are its trivial and sign representations. The action of $\mathfrak{S}_k$ on $(\mathbf{C}^n)^{\otimes k}$ is by permuting tensor factors. We thus see that, in a sense, Sym^k corresponds to the trivial representation of $\mathfrak{S}_k$ and $\bigwedge^k$ to the sign representation. Do the other representations of $\mathfrak{S}_k$ play a role here? Yes!

Recall that a *partition* is a sequence $\lambda = (\lambda_1, \lambda_2, \ldots)$ of non-negative integers that is non-increasing ($\lambda_i \geq \lambda_{i+1}$) and eventually 0 ($\lambda_i = 0$ for $i \gg 0$). The *size* of λ is $|\lambda| = \sum_{i \geq 1} \lambda_i$, and the *length* of λ, denoted $\ell(\lambda)$, is the number of non-zero parts (i.e., the maximum r such that $\lambda_r \neq 0$). We say that λ is a partition of $|\lambda|$.

Conjugacy classes in $\mathfrak{S}_k$ are naturally in bijection with partitions of k, via cycle decompositions. By finite group representation theory, it follows that the number of complex irreducible representations of $\mathfrak{S}_k$ is the number of partitions of k. In fact, there is a natural bijection here: for a partition λ of k, there is an associated representation $\mathbf{M}_\lambda$ of $\mathfrak{S}_k$ called the *Specht module*. Two simple examples:

$$\mathbf{M}_{(k)} = \mathrm{triv}, \qquad \mathbf{M}_{(1^k)} = \mathrm{sgn}\,.$$

A more interesting example: $\mathbf{M}_{(k-1,1)}$ is the standard representation, i.e., the kernel of the augmentation map $\mathbf{C}^k \to \mathbf{C}$.

Now, for a vector space V and a partition λ of k, put

$$\mathbf{S}_\lambda(V) = \mathrm{Hom}_{\mathfrak{S}_k}(\mathbf{M}_\lambda, V^{\otimes k}).$$

Thus

$$\mathbf{S}_{(k)}(V) = \mathrm{Sym}^k(V), \qquad \mathbf{S}_{(1^k)}(V) = \textstyle\bigwedge^k(V).$$

The construction $\mathbf{S}_\lambda(V)$ is functorial in V: if $V \to W$ is a linear map then there is an induced linear map $\mathbf{S}_\lambda(V) \to \mathbf{S}_\lambda(W)$. In particular, $\mathbf{GL}(V)$ acts on $\mathbf{S}_\lambda(V)$, and so $\mathbf{S}_\lambda(\mathbf{C}^n)$ is a representation of $\mathbf{GL}_n$. The functor $\mathbf{S}_\lambda$ is called the *Schur functor* associated to λ. A *polynomial functor* is a direct sum of Schur functors, or a functor isomorphic to such. We allow infinite direct sums, since we want the symmetric algebra $\mathrm{Sym} = \bigoplus_{k \geq 0} \mathrm{Sym}^k$ to be a polynomial functor.

The following key result connects Schur functors and highest weight theory, and demonstrates an intimate connection between the representation theories of symmetric groups and general linear groups.

THEOREM 8.1. *Let λ be a partition and let n be a non-negative integer.*

(a) *If $n < \ell(\lambda)$ then $\mathbf{S}_\lambda(\mathbf{C}^n) = 0$.*
(b) *If $n \geq \ell(\lambda)$ then $\mathbf{S}_\lambda(\mathbf{C}^n)$ is the irreducible polynomial representation of $\mathbf{GL}_n$ with highest weight λ (or really, $(\lambda_1, \ldots, \lambda_n)$).*

Proof. See [FH, §6, §15.5]. □

Exercises

Exercise 8.1. Put $T_k(V) = V^{\otimes k}$. Show that the following conditions on a functor F are equivalent:

(a) F is polynomial.
(b) F is a subfunctor of a direct sum of T_k's.
(c) F is a direct summand of a direct sum of T_k's.

As an application, show that a tensor product of polynomial functors is a polynomial functor.

Exercise 8.2. Let V be a vector space. Show that $\mathbf{S}_{(n-1,1)}(V)$ is naturally isomorphic to the kernel of the multiplication map $\mathrm{Sym}^{n-1}(V) \otimes V \to \mathrm{Sym}^n(V)$.

Exercise 8.3. Let F be a polynomial functor. Define $\ell(F)$ to be the supremum of the set

$$\{\ell(\lambda) \mid \mathbf{S}_\lambda \text{ is a summand of } F\}$$

Suppose that $\ell(F) = n$ is finite (we say that F is *bounded*). Show that the function

$$\{\text{subfunctors of } F\} \to \{\mathbf{GL}_n\text{-subrepresentations of } F(\mathbf{C}^n)\}$$
$$G \mapsto G(\mathbf{C}^n)$$

is a well-defined bijection. (This gives a precise sense in which evaluating on $\mathbf{C}^n$ does not lose information.)

Additional exercises

Exercise 8.4. Let V and W be vector spaces. Show that there are natural isomorphisms

$$\mathrm{Sym}^n(V \oplus W) = \bigoplus_{i+j=n} \mathrm{Sym}^i(V) \otimes \mathrm{Sym}^j(W)$$
$$\mathrm{Sym}^n(V \otimes W) = \bigoplus_{\lambda \vdash n} \mathbf{S}_\lambda(V) \otimes \mathbf{S}_\lambda(W)$$

These are known as the *binomial theorem* and *Cauchy identity*. [Hint: for the Cauchy identity, decompose $V^{\otimes n}$ and $W^{\otimes n}$ using Schur–Weyl duality, then tensor these together and take $\mathfrak{S}_n$-invariants. Second hint: irreducible representations of $\mathfrak{S}_n$ are self-dual.]

Exercise 8.5. In what follows, V denotes a vector space.

(a) Let $S(V) = \mathrm{Sym}(\mathbf{C}^2 \otimes V)$. Show that S is a polynomial functor.
(b) Show that S is bounded, in the sense of Exercise 8.3.
(c) An *ideal* of S is a subfunctor $\mathfrak{a}$ such that $\mathfrak{a}(V)$ is an ideal of $S(V)$ for all V. Show that ideals of S satisfy the ascending chain condition. [Hint: use Exercise 8.3.]

Exercise 8.6. Look at the six-dimensional $(1,1,1)$ weight space inside $(\mathbf{C}^3)^{\otimes 3}$. This has an action of $\mathfrak{S}_3$ by permuting the tensor factors, and a separate action of $\mathfrak{S}_3 \subset \mathbf{GL}_3$. Using either action to identify this weight space with the regular representation of $\mathfrak{S}_3$ on $\mathbf{C}[\mathfrak{S}_3]$, identify the other action.

Exercise 8.7. Consider the Specht representation of $\mathfrak{S}_4$ for the partition $\lambda = (2,2)$, living inside the polynomial ring $\mathbf{C}[x_1, x_2, x_3, x_4]$.

(a) Identify the basis of polynomials associated to standard Young tableaux.
(b) There's a non-standard tableau whose polynomial is $(x_1 - x_4)(x_2 - x_3)$. Express this polynomial in terms of the basis. You may also want to practice by expressing the polynomials associated to other non-standard tableaux in terms of the basis.
(c) Prove that this Specht representation is irreducible.
(d) Let $s = (12) \in \mathfrak{S}_4$. Find an eigenbasis for s. Show that this is a simultaneous eigenbasis for the four *Young-Jucys-Murphy operators*

$$j_1 = 0, \qquad j_2 = (12), \qquad j_3 = (13) + (23), \qquad j_4 = (14) + (24) + (34)$$

in $\mathbf{C}[\mathfrak{S}_4]$. Find the eigenvalues of these YJM operators, and try to relate them to the standard young tableaux of shape λ.
(e) The Specht representation generates an $\mathfrak{S}_4$-invariant ideal $I_\lambda \subset \mathbf{C}[x_1, x_2, x_3, x_4]$, whose vanishing set is an $\mathfrak{S}_4$-invariant algebraic set inside $\mathbf{C}^4$. Describe this vanishing set.

Exercise 8.8. Compute all irreducible representations of $\mathfrak{S}_4$ and their dimensions. For $n \in \{2,3,4\}$, write down the decomposition of $(\mathbf{C}^n)^{\otimes 4}$ given by Schur–Weyl duality. Compute the dimensions of L_λ for each summand in this decomposition, and make sure the overall dimensions add up appropriately.

Exercise 8.9. Let V be a rational $\mathbf{GL}_n$ representation. Prove that $\mathrm{End}_{\mathbf{GL}_n}(V)$ is semisimple. Prove that the multiplicity spaces in V are irreducible modules for $\mathrm{End}_{\mathbf{GL}_n}(V)$. [Hint: What are the irreducible representations of a matrix algebra? Of a product of matrix algebras?]

Notes

The material in this lecture is well-known. Symmetric group representations are discussed in [FH, §4]. Schur functors and polynomial functors are discuss in [FH,

§6] and [Ma, I.A]. All this material is also discussed at length in [SS]. The notation for Specht modules is usually something like $\mathbf{S}^\lambda$; we used a different notation to try and avoid confusion with Schur functors.