Lecture 9

Small subalgebras

Recall that Principle 1.7 states that polynomials of high collective strength behave approximately like independent variables. As independent variables forms a regular sequence, this suggests that polynomials of high collective strength should form a regular sequence. We now prove this:

**Theorem 9.1.** Given integers $d_1, \ldots, d_r$ there is an integer $B = B(d_1, \ldots, d_r)$ with the following property. If $f_1, \ldots, f_r \in k[x_1, \ldots, x_n]$ are homogeneous of degrees $d_1, \ldots, d_r$ and collective strength $> B$ then $f_1, \ldots, f_r$ form a regular sequence.

We emphasize that the key aspect of the theorem is that the quantity $B$ is independent of the number of variables $n$.

**Proof.** We argue as follows:

- Suppose the statement is false. Then for each positive integer $i$ we can find a collection $(f_{1,i}, \ldots, f_{r,i})$ of polynomials (in some number of variables) of strength at least $i$ which does not form a regular sequence. We treat the $f_i$'s as elements of $R_\infty$.
- Let $f_1, \ldots, f_r$ be the elements of $S$ defined by $f_{1,\bullet}, \ldots, f_{r,\bullet}$. As we have seen (Exercise 3.2), $f_1, \ldots, f_r$ have infinite collective strength. (Here we take the index set $I$ for the ultraproduct to be the positive integers.)
- Since $S$ is a polynomial ring (Exercise 3.2), it follows that $f_1, \ldots, f_r$ forms a regular sequence in $S$; in fact, the $f_i$'s can be taken to be “variables” in the isomorphism of $S$ with a polynomial ring.
- By Corollary 6.3, we see that $f_{1,i}, \ldots, f_{r,i}$ forms a regular sequence for all $i$ in some set in the ultrafilter; this is a contradiction.

We now come to a very important result: the existence of small subalgebras. We begin with an informal discussion. Let $f_1, \ldots, f_r$ be homogeneous polynomials. If $f_1, \ldots, f_r$ have “large” collective strength then they form a regular sequence by the above theorem. Otherwise, some homogeneous linear combination can be expressed using a “small” number of elements of lower degrees. We can then apply this same observation to the elements appearing in this expression. Continuing in this manner, we see that $f_1, \ldots, f_r$ can be expressed in terms of elements $g_1, \ldots, g_s$ which have “large” collective strength and with $s$ “small.” The above theorem implies that $g_1, \ldots, g_s$ form a regular sequence. Thus, in any case, $f_1, \ldots, f_r$ belong to a subalgebra generated by a regular sequence of “small” length. This is the small subalgebra.

We now formalize the above discussion:
Theorem 9.2. Fix positive integers \(d_1, \ldots, d_r\). Then there exists an integer \(C = C(d_1, \ldots, d_r)\) with the following property. If \(f_1, \ldots, f_r\) are homogeneous elements of \(k[x_1, \ldots, x_n]\) of degrees \(d_1, \ldots, d_r\) then there exists a homogeneous regular sequence \(g_1, \ldots, g_s\) with \(\deg(g_j) \leq \max(d_1, \ldots, d_r)\) and \(s \leq C\) such that each \(f_i\) belongs to the subalgebra \(k[g_1, \ldots, g_s]\).

Again, the key point is that \(C\) is independent of the number of variables \(n\).

Proof. Let \(B = B(d_1, \ldots, d_r)\) be as in Theorem 9.1. If the collective strength of \(f_1, \ldots, f_r\) is \(> B\) then the \(f_i\)'s form a regular sequence by Theorem 9.1, and so we can take \((g_1, \ldots, g_s) = (f_1, \ldots, f_r)\).

Now suppose that \((f_1, \ldots, f_r)\) have collective strength \(\leq B\). Making a linear change of variables, we may as well assume we have an expression \(f_r = \sum_{i=1}^{B} a_i b_i\) where the \(a_i\) and \(b_i\) are homogeneous of degree \(< d_r\). Let \(d'_i = \deg(a_i)\) and \(d''_i = \deg(b_i)\). The tuple

\[ e = (d_1, \ldots, d_{r-1}, d'_1, \ldots, d'_B, d''_1, \ldots, d''_B) \]

is smaller than the tuple \(d = (d_1, \ldots, d_r)\), in the sense that \(d_r\) has been replaced by a list of numbers that are all strictly smaller than it. By induction (see Exercise ??), we can therefore assume that result holds in degree \(e\). Thus there is a homogeneous regular sequence \(g_1, \ldots, g_s\) with \(\deg(g_j) \leq \max(e) \leq \max(d)\) and \(s \leq C(e)\) such that \(f_1, \ldots, f_{r-1}\) and the \(a's\) and \(b's\) belong to \(k[g_1, \ldots, g_s]\). Of course, this implies that \(f_r\) also belongs to \(k[g_1, \ldots, g_s]\). \(\square\)

Remark 9.3. The proof shows that we can take \(C(d)\) to be the maximum of \(B(d)\) and \(C(e)\) as \(e\) varies over all ways of replacing one \(d_i\) with a list of at most \(2B(d)\) strictly smaller numbers (really \(B(d)\) pairs of positive integers summing to \(d_i\)).

Exercises

Exercise 9.1. Show that if \(f_1\) and \(f_2\) are linearly independent irreducible homogeneous polynomials then \(f_1, f_2\) is a regular sequence. Conclude that we can take \(B(d_1, d_2) = 2\) in Theorem 9.1.

Exercise 9.2. A priori, the bound \(B(d_1, \ldots, d_r)\) also depends on the field \(k\). Show that it is in fact independent of \(k\), at least if we work in characteristic 0. [Hint: consider the graded ultraproduct of rings \(k_i[x_1, x_2, \ldots]\) for varying fields \(k_i\). Show that this is a polynomial ring, and then carry out our same arguments.]

Notes

Theorem 9.1 was originally proved by Ananyan–Hochster \([AH, Theorems A]\). The proof given here is different from the original one, and comes from \([ESS2, Theorem 4.11]\). The existence of small subalgebras (Theorem 9.2) was also originally by Ananyan–Hochster \([AH, Theorems B]\). The proof given here is essentially the same.
as the one given there, and in [ESS2, Theorem 4.12]. For more expository treatments, see [ESS3, §6] (for regular sequences) and [ESS3, §8] (for small subalgebras).

Some explicit results are known for the quantity $B$ in Theorem 9.1; see [Ch].