LECTURE 9

Small subalgebras

Recall that Principle 1.7 states that polynomials of high collective strength behave approximately like independent variables. As independent variables forms a regular sequence, this suggests that polynomials of high collective strength should form a regular sequence. We now prove this:

THEOREM 9.1. *Given integers $d_1, \ldots, d_r$ there is an integer $B = B(d_1, \ldots, d_r)$ with the with the following property. If $f_1, \ldots, f_r \in k[x_1, \ldots, x_n]$ are homogeneous of degrees $d_1, \ldots, d_r$ and collective strength $> B$ then $f_1, \ldots, f_r$ form a regular sequence.*

We emphasize that the key aspect of the theorem is that the quantity B is independent of the number of variables n.

Proof. We argue as follows:

- Suppose the statement is false. Then for each positive integer i we can find a collection $(f_{1,i}, \ldots, f_{r,i})$ of polynomials (in some number of variables) of strength at least i which does not form a regular sequence. We treat the f's as elements of R_∞.
- Let $f_1, \ldots, f_r$ be the elements of $\mathbf{S}$ defined by $f_{1,\bullet}, \ldots, f_{r,\bullet}$. As we have seen (Exercise 3.2), $f_1, \ldots, f_r$ have infinite collective strength. (Here we take the index set I for the ultraproduct to be the positive integers.)
- Since $\mathbf{S}$ is a polynomial ring (Exercise 3.2), it follows that $f_1, \ldots, f_r$ forms a regular sequence in $\mathbf{S}$; in fact, the f_i's can be taken to be "variables" in the isomorphism of $\mathbf{S}$ with a polynomial ring.
- By Corollary 6.3, we see that $f_{1,i}, \ldots, f_{r,i}$ forms a regular sequence for all i in some set in the ultrafilter; this is a contradiction. □

We now come to a very important result: the existence of small subalgebras. We begin with an informal discussion. Let $f_1, \ldots, f_r$ be homogeneous polynomials. If $f_1, \ldots, f_r$ have "large" collective strength then they form a regular sequence by the above theorem. Otherwise, some homogeneous linear combination can be expressed using a "small" number of elements of lower degrees. We can then apply this same observation to the elements appearing in this expression. Continuing in this manner, we see that $f_1, \ldots, f_r$ can be expressed in terms of elements $g_1, \ldots, g_s$ which have "large" collective strength and with s "small." The above theorem implies that $g_1, \ldots, g_s$ form a regular sequence. Thus, in any case, $f_1, \ldots, f_r$ belong to a subalgebra generated by a regular sequence of "small" length. This is the small subalgebra.

We now formalize the above discussion:

Theorem 9.2. *Fix postive integers $d_1, \ldots, d_r$. Then there exists an integer $C = C(d_1, \ldots, d_r)$ with the following property. If $f_1, \ldots, f_r$ are homogeneous elements of $k[x_1, \ldots, x_n]$ of degrees $d_1, \ldots, d_r$ then there exists a homogeneous regular sequence $g_1, \ldots, g_s$ with $\deg(g_j) \le \max(d_1, \ldots, d_r)$ and $s \le C$ such that each f_i belongs to the subalgebra $k[g_1, \ldots, g_s]$.*

Again, the key point is that C is independent of the number of variables n.

Proof. Let $B = B(d_1, \ldots, d_r)$ be as in Theorem 9.1. If the collective strength of $f_1, \ldots, f_r$ is $> B$ then the f_i's form a regular sequence by Theorem 9.1, and so we can take $(g_1, \ldots, g_s) = (f_1, \ldots, f_r)$.

Now suppose that $(f_1, \ldots, f_r)$ have collective strength $\le B$. Making a linear change of variables, we may as well assume we have an expression $f_r = \sum_{i=1}^{B} a_i b_i$ where the a_i and b_i are homogeneous of degree $< d_r$. Let $d'_i = \deg(a_i)$ and $d''_i = \deg(b_i)$. The tuple

$$\mathbf{e} = (d_1, \ldots, d_{r-1}, d'_1, \ldots, d'_B, d''_1, \ldots, d''_B)$$

is smaller than the tuple $\mathbf{d} = (d_1, \ldots, d_r)$, in the sense that d_r has been replaced by a list of numbers that are all strictly smaller than it. By induction (see Exercise **??**), we can therefore assume that result holds in degree $\mathbf{e}$. Thus there is a homogeneous regular sequence $g_1, \ldots, g_s$ with $\deg(g_j) \le \max(\mathbf{e}) \le \max(\mathbf{d})$ and $s \le C(\mathbf{e})$ such that $f_1, \ldots, f_{r-1}$ and the a's and b's belong to $k[g_1, \ldots, g_s]$. Of course, this implies that f_r also belongs to $k[g_1, \ldots, g_s]$. □

Remark 9.3. The proof shows that we can take $C(\mathbf{d})$ to be the maximum of $B(\mathbf{d})$ and $C(\mathbf{e})$ as $\mathbf{e}$ varies over all ways of replacing one d_i with a list of at most $2B(\mathbf{d})$ strictly smaller numbers (really $B(\mathbf{d})$ pairs of positive integers summing to d_i).

Exercises

Exercise 9.1. Show that if f_1 and f_2 are linearly independent irreducible homogeneous polynomials then f_1, f_2 is a regular sequence. Conclude that we can take $B(d_1, d_2) = 2$ in Theorem 9.1.

Exercise 9.2. A priori, the bound $B(d_1, \ldots, d_r)$ also depends on the field k. Show that it is in fact independent of k, at least if we work in characteristic 0. [Hint: consider the graded ultraproduct of rings $k_i[x_1, x_2, \ldots]$ for varying fields k_i. Show that this is a polynomial ring, and then carry out our same arguments.]

Notes

Theorem 9.1 was originally proved by Ananyan–Hochster [AH, Theorems A]. The proof given here is different from the original one, and comes from [ESS2, Theorem 4.11]. The existence of small subalgebras (Theorem 9.2) was also originally by Ananyan–Hochster [AH, Theorems B]. The proof given here is essentially the same as the one given there, and in [ESS2, Theorem 4.12]. For more expository treatments, see [ESS3, §6] (for regular sequences) and [ESS3, §8] (for small subalgebras).

Some explicit results are known for the quantity B in Theorem 9.1; see [Ch].