WARTHOG 2022

Infinite-dimensional methods in commutative algebra

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Contents

Foreword

WARTHOG (Workshop on Algebra and Representation Theory, Held on Oregonian Grounds) is an annual workshop held at the University of Oregon, and currently organized by Ben Elias and Nick Proudfoot. This year[1] (2022), I gave the workshop on topics related to infinite dimensional methods that have recently been used in commutative algebra. The workshop began on Sunday, July 26 and ended on Thursday, July 30. There were about 40 participants.

I had a lot of help running the workshop: Ben and Nick handled the logistics, and helped a lot with the math too. Additionally, Karthik Ganapathy, Bobby Laudone, and Eric Ramos were present to offer their expertise. It is a pleasure to thank all of them.

I wrote these notes (mostly) before the school. The lectures at the school were then (mostly) based on these notes. I gave all lectures except the following ones:

- Lecture 3: Eric
- Lectures 4 and 9: Karthik
- Lectures 6 and 8: Ben
- Lecture 12: Bobby
- Lecture 18: Alessandro Danelon

These lectures (especially Ben's) diverged from the notes somewhat more.

The three lectures on representation theory (Lectures 6, 8, 11) were placed somewhat randomly. This was deliberate: the hope was that by spreading this material out it would give the students more time to absorb it.

WARTHOG places heavy emphasis on exercises. After each lecture (except the final one) there was an exercise session, typically lasting 40–60 minutes. The exercises from the school are included in these notes.

More information can be found at the webpage:

https://pages.uoregon.edu/belias/WARTHOG/infcomm/index.html

[1]In fact, I was supposed to give the school in 2020, but it was delayed two years due to COVID.

Overview

Around the year 2000, Mike Stillman formulated the following conjecture: the projective dimension of a homogeneous ideal in a polynomial ring can be bounded in terms of the number and degrees of the generators, and independent of the number of variables. Stillman's conjecture was proved in 2016 by Ananyan and Hochster. Building on their ideas, Erman, Sam, and Snowden gave two new proofs of the conjecture in 2018, both relying on infinite dimensional methods. Shortly thereafter, Draisma, Lasoń, and Leykin gave a fourth proof using related ideas. The purpose of this workshop is to explain the various tools that go into the proofs of Erman, Sam, and Snowden. Of course, the proofs themselves are covered too!

First proof

We now outline the first proof. This is definitely the shortest and easiest route to the conjecture. Let $\mathbf{S}$ be the "graded ultraproduct" of the polynomial rings. This is the primary infinite dimensional object employed in this proof. The key observation is that $\mathbf{S}$ is abstractly a polynomial ring (in uncountably many variables). This allows us to do all sorts of commutative algebra in $\mathbf{S}$: for example, it implies that a finitely generated ideal of $\mathbf{S}$ has finite height.

Using this feature of $\mathbf{S}$, we prove an important theorem of Ananyan and Hochster: a tuple of polynomials of high "collective strength" forms a regular sequence. The basic idea is as follows. Let $f_{1,i}, \ldots, f_{r,i}$ be a sequence of tuples with collective strength tending to infinity. This defines a tuple $f_1, \ldots, f_r$ in the ultraproduct ring $\mathbf{S}$, which is easily seen to have infinite collective strength. The fact that $\mathbf{S}$ is a polynomial ring implies that $f_1, \ldots, f_r$ is a regular sequence. We then argue that this property transfers back to $f_{1,i}, \ldots, f_{r,i}$ for many indices i.

The above result enables one to construct the "small subalgebras" of Ananyan and Hochster. Once one has these, Stillman's conjecture follows easily.

Second proof

We now discuss the second proof. This is more demanding, but also much more flexible. We work over the complex numbers. Fix positive integer $d_1, \ldots, d_r$; these will be the degrees of the generators of the ideal. Put

$$X_n = \mathrm{Sym}^{d_1}(\mathbf{C}^n) \times \cdots \times \mathrm{Sym}^{d_r}(\mathbf{C}^n).$$

Here $\mathrm{Sym}^d(\mathbf{C}^n)$ is the space of homogeneous degree d polynomials in n variables. We regard X_n as a parameter space for (generators of) ideals: to a point $x = (f_1, \ldots, f_r)$ of X_n we associate the ideal $\mathfrak{a}_x$ of $\mathbf{C}[x_1, \ldots, x_n]$ generated by the f's.

In Stillman's conjecture, the number of variables is irrelevant. It is therefore natural to consider a space analogous to X_n, but where we do not fix the number of variables. The most obvious way to do this is to take the direct limit (union) of the X_n's. However, this is not a scheme: it is an ind-scheme. Instead, we work with the inverse limit X. This is an affine scheme, and the primary infinite dimensional object used in the second proof.

Since X is infinite dimensional, we need some extra structure to get a handle on it. Fortunately, this is available: the infinite general linear group $\mathbf{GL}$ acts on X. In fact, X is what we call a $\mathbf{GL}$-*variety*. The most important property of $\mathbf{GL}$-varieties is Draisma's theorem: they are $\mathbf{GL}$-noetherian, that is, a descending chain of $\mathbf{GL}$-stable closed subsets stabilizes.

To a point $x \in X$, we have an ideal $\mathfrak{a}_x$ in the graded inverse limit $\mathbf{R}$ of the polynomial rings. To prove Stillman's conjecture, it is enough to show that these ideals have bounded projective dimension. Let $Z_n \subset X$ be the locus where the projective dimension is $\geq n$. These form a descending chain of $\mathbf{GL}$-stable subsets. (Note that $\mathbf{GL}$ simply makes a linear change of variables, which does not affect the projective dimension.) If the Z_n's were closed then Draisma's theorem would imply the desired result. In fact, the Z_n's are not necessarily closed, but one can still basically apply this reasoning.

To make the above proof work, one has to carry out some commutative algebra in $\mathbf{R}$. Fortunately, it too is abstractly a polynomial ring—this is proved just like for $\mathbf{S}$.

Outline of lectures

We briefly summarize the contents of the lectures:

- On Sunday we introduce Stillman's conjecture, review ultraproducts, define the two rings $\mathbf{R}$ and $\mathbf{S}$, and prove that they are polynomial rings.
- On Monday and Tuesday we give the first proof of Stillman's conjecture.
- On Wednesday we introduce $\mathbf{GL}$-varieties and prove Draisma's theorem.
- (Mixed in among the lectures on Monday and Wednesday are three lectures on the representation theory of $\mathbf{GL}_n$. This background material is needed to work with $\mathbf{GL}$-varieties.)
- The first lecture on Thursday sketches the second proof of Stillman's conjecture, while the second lecture shows how one can get even stronger results with this approach.
- The two final lectures on Thursday go in a somewhat different direction, and look at the internal structure of elements of $\mathbf{R}$ of infinite strength.

LECTURE 1

Stillman's conjecture

Let k be a field and let $R_n = k[x_1, \dots, x_n]$ be the n-variable polynomial ring. Recall the famous Hilbert syzygy theorem:

THEOREM 1.1. *Any R_n-module has projective dimension $\leq n$.*

The R_n-module $k = R_n/(x_1, \dots, x_n)$ has projective dimension n, which shows that the theorem is optimal. More generally, if $f_1, \dots, f_r$ is a regular sequence in R_n then $R_n/(f_1, \dots, f_r)$ has projective dimension r. One might therefore expect that the projective dimension of an ideal generated by r elements would have projective dimension at most r. This is too simplistic, but only slightly: Mike Stillman formulated the following conjecture around the year 2000.

CONJECTURE 1.2. *The projective dimension of $R_n/\mathfrak{a}$, for a homogeneous ideal $\mathfrak{a}$ of R_n, can be bounded in terms of the number and degrees of generators $\mathfrak{a}$ (and independent of n). In other words, given $d_1, \dots, d_r$ there is $N = N(d_1, \dots, d_r)$ such that $\operatorname{pdim}_{R_n}(R_n/\mathfrak{a}) \leq N$ whenever $\mathfrak{a} \subset R_n$ is generated by homogeneous polynomials $f_1, \dots, f_r$ of degrees $d_1, \dots, d_r$.*

REMARK 1.3. For any n, there are examples where $R/(f_1, f_2, f_3)$ has projective dimension n [Br, Bu, Ko]. Thus the dependence on degree is necessary in Stillman's conjecture. Also, for fixed d and $n \gg d$, there are examples of $R/(f_1, f_2, f_3)$ of projective dimension at least $d^{(\sqrt{d}-1)/2}$ [BM$^+$]. Thus the bound N in the conjecture must be rather fast-growing.

Thie conjecture is now a theorem, first proved by Ananyan and Hochster in 2016. We'll give two proofs of the conjecture this week following the paper of Erman, Sam, and Snowden. In the rest of this lecture, we'll introduce some key concepts. The most important of these is the following definition, due to Ananyan–Hochster:

DEFINITION 1.4. A homogeneous element f of a graded ring has *strength* $\leq s$ if there is an expression $f = \sum_{i=1}^{s} g_i h_i$ where g_i and h_i are homogeneous elements of positive degree[2]. If there is no such expression for any s then f has infinite strength.

EXAMPLE 1.5. A polynomial has strength 0 if and only if it is zero, and strength 1 if and only if it is non-zero and not irreducible. The polynomial $x_1^2 + x_2^2$ clearly has strength ≤ 2, but it has strength 1 if $-1 = i^2$ is a square in k, since

$$x_1^2 + x_2^2 = (x_1 + ix_2)(x_1 - ix_2).$$

The following generalization of strength will also be helpful:

[2]This definition is off by one from the original definition of Ananyan–Hochster.

Definition 1.6. Let $\{f_i\}_{i \in I}$ be a family of homogeneous elements in a graded k-algebra. The *collective strength* of the f_i's is the minimal strength of a non-trivial homogeneous k-linear combination of the f_i's. (Note that f_i's of different degree do not interact in this definition.)

The proof of Ananyan–Hochster actually revealed the following general principle that can be used to predict statements similar to Stillman's conjecture, and will guide us towards the proof of Stillman's conjecture that we present:

Principle 1.7. *If $f_1, \ldots, f_r$ are homogeneous polynomials of high collective strength then $f_1, \ldots, f_r$ behave approximately like r independent variables.*

We briefly explain how Stillman's conjecture is a consequence of an instance of this principle; we elaborate on the details in later lectures. Let $f_1, \ldots, f_r \in R_n$ be given, and let $\mathfrak{a}$ be the ideal they generate. If $f_1, \ldots, f_r$ have sufficiently "large" collective strength, then they form a regular sequence (by the Principle 1.7), and so $\operatorname{pdim}_{R_n}(R_n/\mathfrak{a}) = r$. On the other hand, if $f_1, \ldots, f_r$ have "small" collective strength, some linear combination can be written in terms of lower degree polynomials. By repeating this procedure, we eventually express $f_1, \ldots, f_r$ in terms of polynomials $g_1, \ldots, g_s$ that have high collective strength. It is not difficult to bound s in terms of the degrees of the f_i's. Let S be the subalgebra of R_n generated by the g_i's; this is isomorphic to a polynomial ring in s variables, since regular sequences are algebraically independent. Letting $\mathfrak{b}$ be the ideal of S generated by the f_i's, we have $\operatorname{pdim}_S(S/\mathfrak{b}) \leq s$ by Hilbert's theorem. Since the g_i's are a regular sequence, the inclusion $S \to R$ is flat, and this allows us to deduce that $\operatorname{pdim}_{R_n}(R_n/\mathfrak{a}) \leq s$ as well.

Exercises

Exercise 1.1. Determine the strength of $x_1^2 + \cdots + x_n^2$ over the complex numbers.

Exercise 1.2. Let R be any graded k-algebra and let R_+ be the ideal of positive degree elements in R.

(a) Show that a homogeneous element has finite strength if and only if it belongs to R_+^2. In particular, the finite strength elements form an ideal.
(b) Show that a collection of homogeneous elements has infinite collective strength if and only if it maps to a k-linearly independent set in R_+/R_+^2.
(c) Show that the homogeneous elements of R of infinite strength generate R as a ring. Even better, show that is a family of elements $\{f_i\}_{i \in I}$ of infinite collective strength that generates R.

Additional exercises

Exercise 1.3. Let $f \in k[x_1, \ldots, x_n]$ be homogeneous of positive degree d with k a field of characteristic 0, and let J be the ideal generated by the partial derivatives of f.

(a) Prove *Euler's formula*: $f = \frac{1}{d}\sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i}$.
(b) If f has strength $\le s$ show that J is contained in an ideal generated by $\le 2s$ elements.
(c) Conversely, if J is contained in an ideal generated by $\le s$ elements show that f has strength $\le s$.

Exercise 1.4. Show that $k = R_n/(x_1, \dots, x_n)$ has projective dimension n as an R_n-module.

Exercise 1.5. Let $V_1, \dots, V_n$ be vector spaces. Let $x \in V_1 \otimes \cdots \otimes V_n$. Recall that x is a *pure tensor* if $x = v_1 \otimes \cdots \otimes v_n$ for some $(v_1, \dots, v_n) \in V_1 \times \cdots \times V_n$. We say that x has *tensor rank* $\le r$ if x is a sum of $\le r$ pure tensors.

(a) Show that the tensor rank of x is at most $\prod_{i=1}^{n} \dim V_i$, assuming each V_i is finite dimensional.
(b) Let V be finite dimensional. Recall there is a natural isomorphism $\mathrm{End}(V) = V \otimes V^*$. Show that the tensor rank of an element of $V \otimes V^*$ coincides with its usual rank as an endomorphism of V.
(c) Again, let V be finite dimensional, and suppose k has characteristic $\neq 2$. We can identify the space $\mathrm{Sym}^2(V)$ of degree 2 polynomials in V with a subspace of $V^{\otimes 2}$. How do strength and tensor rank compare?

Exercise 1.6. Show that the polynomial $\sum_{i=1}^{n} x_i y_i z_i$ has strength n. (This is not so easy; see [DES, §4] for a proof.)

Notes

Stillman posed his conjecture around the year 2000. It first appeared in print in [En]; see also [PS]. The conjecture was first proved by Ananyan and Hochster [AH] in 2016. Using ideas from this proof, Erman, Sam, and Snowden [ESS2] gave two new and simpler proofs in 2018. Shortly thereafter, Draisma, Lasoń, and Leykin [DLL] gave a fourth proof using related ideas (but with some new elements). An expository account of the work appears in [ESS3].

LECTURE 2

Ultraproducts

Principle 1.7 is an asymptotic statement: as collective strength increases, polynomials behave more and more like independent variables. This suggests that we might be able to pass to a limiting situation where we can say that things of infinite collective strength behave exactly like independent variables. There are two ways to create a limit of polynomial rings that will be relevant to us: inverse limits and ultraproducts. Inverse limits are reasonably well-known. Ultraproducts, on the other hand, are a bit more obscure (and much more subtle). We therefore review the basics of ultraproducts in this lecture.

Let $\mathcal{I}$ be a set, let x be an element of $\mathcal{I}$, and let $\mathcal{F} = \mathcal{F}_x$ be the collection of all subsets of $\mathcal{I}$ containing x. The collection $\mathcal{F}$ satisfies the following conditions:

(a) The empty set is not in $\mathcal{F}$.
(b) Given $A \subset B$ with $A \in \mathcal{F}$ we have $B \in \mathcal{F}$.
(c) If $A, B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$.
(d) If A is any subset of $\mathcal{I}$ then either A or $\mathcal{I} \setminus A$ belongs to $\mathcal{F}$.

This motivates the following definition:

DEFINITION 2.1. An *ultrafilter* on a set $\mathcal{I}$ is a collection $\mathcal{F}$ of subsets of $\mathcal{I}$ satisfying conditions (a)–(d) above.

Thus $\mathcal{F}_x$ is an example of an ultrafilter; an ultrafilter of this form is called *principal*. A natural question is if non-principal ultrafilters exist. This is answered in the following proposition:

PROPOSITION 2.2. *A set $\mathcal{I}$ admits a non-principal ultrafilter if and only if it is infinite.*

Proof. Let $\mathcal{I}$ be an infinite set, and consider the ring $R = \prod_{i \in I} \mathbf{F}_2$, where $\mathbf{F}_2$ is the field with two elements. For a subset A of $\mathcal{I}$, let 0_A be the element of R having 0's in the A coordinates and 1's in the remaining coordinates. We have a bijection

$$\{\text{ultrafilters on } \mathcal{I}\} \leftrightarrow \{\text{maximal ideals of } R\}$$

as follows: the maximal ideal corresponding to the ultrafilter $\mathcal{F}$ is $\{0_A \mid A \in \mathcal{F}\}$. The proof is left as an exercise (Exercise 2.6).

Given an element $x \in \mathcal{I}$ we have a ring homomorphism $\pi_x \colon R \to \mathbf{F}_2$ by projecting onto the x coordinate. The kernel $\mathfrak{m}_x$ of π_x consists of all elements whose x coordinate is 0. We thus see that $\mathfrak{m}_x$ corresponds to $\mathcal{F}_x$ under the above bijection. To construct a non-principal ultrafilter on $\mathcal{I}$, it thus suffices to construct a maximal ideal of R that's not of the form $\mathfrak{m}_x$.

Let $\mathfrak{a} \subset R$ be the set of all elements whose coordinates have finitely many 1's. This is clearly an ideal of R. We have $\mathfrak{a} \not\subset \mathfrak{m}_x$ for any x, as $\mathfrak{a}$ contains the element that has a 1 in the x coordinate and 0 in all other coordinates. By Zorn's lemma, $\mathfrak{a}$ is contained in some maximal ideal $\mathfrak{m}$. Thus $\mathfrak{m}$ is a maximal ideal that is not one of the $\mathfrak{m}_x$'s, and therefore yields a non-principal ultrafilter.

Now suppose $\mathcal{I}$ is finite. We must show that every ultrafilter is principal. We leave this as an exercise (Exercise 2.3). $\square$

Fix a non-principal ultrafilter $\mathcal{F}$ on $\mathcal{I}$. Let $\{X_i\}_{i \in \mathcal{I}}$ be a family of sets. We define an equivalence relation $\sim$ on the product $\prod_{i \in \mathcal{I}} X_i$ as follows: $x \sim y$ if there is a set $A \in \mathcal{F}$ such that $x_i = y_i$ for all $i \in A$. We now come to the main definition of this lecture:

Definition 2.3. The *ultraproduct* of the X_i's (with respect to $\mathcal{F}$) is $(\prod_{i \in \mathcal{I}} X_i)/\sim$.

One can think of the ultraproduct X^* of the X_i's as a kind of limit of the X_i's. Given a sequence $(x_i)_{i \in \mathcal{I}}$ with $x_i \in X_i$, there is tautologically a "limit" x^* in X^*: indeed, (x_i) is simply an element of the product, and thus its equivalence class is an element of X^*. Thus, in a sense, all sequences have a limit in the ultraproduct.

One very useful feature of ultraproducts is that if the X_i's have some kind of algebraic structure (group, ring, field, etc.) then X^* will naturally carry this structure as well; see Exercise 2.1.

Remark 2.4. Psychologically, it is useful to think of $\mathcal{F}$ as specifying a hypothetical point $*$ of $\mathcal{I}$. The condition $x \sim y$ can then be thought of as saying that $x_i = y_i$ holds in a neighborhood of $*$.

Exercises

Exercise 2.1. Let $\mathcal{I}$ be a set equipped with a non-principal ultrafilter $\mathcal{F}$. Let $\{K_i\}_{i \in \mathcal{I}}$ be a family of fields, and let K^* be their ultraproduct.

(a) Show that K^* is a field under coordinatewise operations. Pay special attention to reciprocals.
(b) Suppose that $\mathcal{I}$ is the set of prime numbers and $K_p = \mathbf{F}_p$. Show that K^* has characteristic 0.

Now suppose that for each $i \in \mathcal{I}$ we have a K_i-vector space V_i. Let V^* be the ultraproduct of the V_i's.

(c) Show that V^* is naturally a K^*-vector space.
(d) Suppose $\dim_{K_i}(V_i) = n$ is finite for all $i \in \mathcal{I}$. Show that $\dim_{K^*}(V^*) = n$.

Exercise 2.2. Let $\mathcal{I}$ be a countable set equipped with a non-principal ultrafilter $\mathcal{F}$. Let $\mathbf{R}^*$ be the ultrapower of the real numbers $\mathbf{R}$. (Ultrapower is just the case where the X_i's are all the same set.) Identify $a \in \mathbf{R}$ with the element $(a, a, \ldots)$ in $\mathbf{R}^*$.

(a) Show that $\mathbf{R}^*$ carries a natural total order.

(b) Show that there is an element $\epsilon \in \mathbf{R}^*$ such that $0 < \epsilon < a$ for all positive $a \in \mathbf{R}$.

An element ϵ as in (b) is called an *infinitesimal*. These elements form the basis of non-standard analysis.

Additional exercises

Exercise 2.3. Show that any ultrafilter on a finite set is principal.

Exercise 2.4. Show that a non-principal ultrafilter contains all cofinite sets.

Exercise 2.5. Let $\mathcal{I}$ be the set of prime numbers, for $p \in \mathcal{I}$ let $K_p = \mathbf{F}_p$, and let K^* be the ultraproduct of the K_p's with respect to an ultrafilter $\mathcal{F}$ on $\mathcal{I}$.

(a) Show that one can choose $\mathcal{F}$ so that -1 is a square in K^*.
(b) Show that one can choose $\mathcal{F}$ so that -1 is not a square in K^*.
(c) Try to figure out various possibilities for the behavior of algebraic numbers in K^*. E.g., is it possible for K^* to contain no irrational algebraic numbers?

Exercise 2.6. Prove the correspondence between ultrafilters and maximal ideals from Proposition 2.2.

Notes

We proved Proposition 2.2 (existence of non-principal ultrafilters) using the axiom of choice (in the form of Zorn's lemma). As one might expect, Proposition 2.2 is independent of ZF set theory. In fact, Proposition 2.2 is significantly weaker than the axiom of choice: ZF together with Proposition 2.2 cannot even prove countable choice.

Lecture 3

Limits of polynomial rings

As we said in the previous lecture, one might hope to pass to a limiting situation where Principle 1.7 becomes an exact statement. In this lecture, we introduce the two limiting objects that we'll use to realize this dream.

Recall that $R_n = k[x_1, \dots, x_n]$. There is a natural map $R_{n+1} \to R_n$ that kills x_{n+1} and maps x_i to x_i for $1 \le i \le n$. Thus the R_n's form an inverse system. Our first limiting object is the following:

Definition 3.1. The *inverse limit ring* $\mathbf{R}$ is the inverse limit of the rings R_n *in the category of graded rings*.

In the above definition, we have emphasized the inverse limit takes place in the category of graded rings. This means that $\mathbf{R}$ is itself a graded ring, and that its degree d piece $\mathbf{R}_d$ is the inverse limit of the degree d pieces of the R_n's. Explicitly, a homogeneous degree d element of $\mathbf{R}$ is a formal k-linear combination of degree d monomials in the variables $\{x_i\}_{i\ge 1}$. For example, $\sum_{i\ge 1} x_i$ is a degree 1 element of $\mathbf{R}$ and $\sum_{i\ge 1} x_i^2$ is a degree 2 element.

In the polynomial ring R_n, the only elements of infinite strength are linear. However, the inverse limit ring $\mathbf{R}$ has many elements of infinite strength. For example, $\sum_{i\ge 1} x_i^n$ has infinite strength provided n is not divisible by the characteristic of k (Exercise 3.1). In particular, the limiting form of Principle 1.7 (which concerns elements of infinite collective strength) is an interesting statement in $\mathbf{R}$ (which we will consider in the subsequent lecture).

To prove Stillman's conjecture, we want to use the following style of argument: let $\{f_i\}$ be a sequence of polynomials of increasing strength; pass to a limit f that has infinite strength; use the limiting form of Principle 1.7 to say something about f; pass this information back to the f_i's. Unfortunately, if $\{f_i\}$ is an arbitrary sequence of polynomials, it won't necessarily have any kind of sensible limit in $\mathbf{R}$ (though see Exercise 3.3). However, we saw in the previous lecture that "all limits exist" in ultraproducts, so this suggests that we should be using an ultraproduct construction.

We now introduce the precise ultraproduct we want to use. Let $R_\infty = k[x_1, x_2, \dots]$ be the polynomial ring in variables $\{x_i\}_{i\ge 1}$. Fix an infinite set $\mathcal{I}$ and a non-principal ultrafilter $\mathcal{F}$ on $\mathcal{I}$. (In practice, one can always take $\mathcal{I}$ to be the set of natural numbers, but we avoid making a specific choice for now.)

Definition 3.2. The *ultraproduct ring* $\mathbf{S}$ is the *graded* ultrapower of R_∞.

We need to clarify what the word "graded" is doing in the above definition. This means that $\mathbf{S}$ is a graded ring, and that its degree d piece $\mathbf{S}_d$ is the ultrapower of the degree d piece of R_∞. Note in particular that $\mathbf{S}_0 = k^*$ is the ultrapower of the coefficient field k, which will be much larger than k (unless k is finite).

Remark 3.3. One might wonder: why are we taking the ultrapower of R_∞ instead of taking the ultraproduct of the R_n's? In fact, it doesn't really matter which we use, but I find using R_∞ more natural. With the way we have set things up, $\mathcal{I}$ is an arbitrary index set, and any sequence of polynomials in the variables $\{x_i\}_{i\ge 1}$ indexed by $\mathcal{I}$ will have a limit in $\mathbf{S}$. If one wanted to use the ultraproduct of the family $\{R_n\}_{n\in\mathbf{N}}$, this would involve choosing an ultrafilter on $\mathbf{N}$, and one could only consider sequences $\{f_n\}$ where f_n uses the first n variables; this seems a bit arbitrary.

Exercises

Exercise 3.1. Work over the complex numbers, and let d be a positive integer. Show that $\sum_{i\ge 1} x_i^d$ has infinite strength in $\mathbf{R}$. [Hint: show that if this element had strength $s < \infty$ then any degree d polynomial in n variables would have strength $\le s$ and obtain a contradiction.]

Exercise 3.2. Let $f_i \in k[x_1, x_2, \ldots]$ be homogeneous of degree d and let $f \in \mathbf{S}$ be the corresponding element. Show that f has finite strength if and only if there is some set J in the ultrafilter such that the elements $\{f_j\}_{j\in J}$ have bounded strength. Formulate and prove a similar result for collective strength of a finite collection of elements.

Additional exercises

Exercise 3.3. Suppose that k is a finite field. For each $n \ge 1$, let f_n be a homogeneous degree d polynomial in $k[x_1, \ldots, x_n]$.

(a) Show that there is a homogeneous degree d element f in $\mathbf{R}$ such that f maps to f_n for infinitely many n. [Hint: $\mathbf{R}_d$ is a compact topological space (how?).]
(b) Suppose that the strength of the f_n's goes to infinity. Show that f has infinite strength.
(c) Suppose that the f_n's have bounded strength. Show that f has finite strength.

Exercise 3.4. Suppose $\operatorname{char}(k) \ne 2$. Let $f = \sum_{1\le i\le j} a_{i,j} x_i x_j$ be a degree two element of $\mathbf{R}$.

(a) Show that f has strength 1 if and only if $a_{i,j}a_{k,\ell} = a_{i,\ell}a_{j,k}$ (with the convention $a_{i,j} = a_{j,i}$).
(b) Show that the condition "f has strength $\le s$" is equivalent to a system of polynomial equations in the coefficients $a_{i,j}$.

Exercise 3.5. Let $\mathbf{R}_{k^*}$ be the graded inverse limit of the rings $k^*[x_1, \ldots, x_n]$.

(a) Construct a natrual homomorphism of k^*-algebras $\varphi \colon \mathbf{S} \to \mathbf{R}_{k^*}$. [Hint: given $f \in \mathbf{S}$ define $\varphi(f)$ by specifying the coefficient of each monomial.]

(b) Determine if φ is injective or surjective (or both or neither).

Exercise 3.6. Write down an element of the inverse limit of the R_n's in the category of (ungraded) rings that does not belong to $\mathbf{R}$.

Notes

The rings $\mathbf{R}$ and $\mathbf{S}$ were introduced (in the context of Stillman's conjecture) in [ESS2, §5] and [ESS2, §4]. A more expository treatment of these objects can be found in [ESS3, §4] and [ESS2, §5].

We defined $\mathbf{S}$ as the graded ultrapower of $k[x_1, x_2, \ldots]$. More generally, one could start with a family of fields $\{k_i\}_{i \in \mathcal{I}}$ and define $\mathbf{S}$ as the graded ultraproduct of the rings $k_i[x_1, x_2, \ldots]$. This allows one to prove statements (like Stillman's conjecture) uniformly in the coefficient field. We have opted for the less general set-up in these lectures just to keep things a bit more simple.

Lecture 4

Big polynomial rings

Principle 1.7 suggests that a set of elements of $\mathbf{R}$ of infinite collective strength should be algebraically independent. On the other hand, a maximal such set generates $\mathbf{R}$ (Exercise 1.2). This suggests that $\mathbf{R}$ should abstractly be a polynomial ring. How can we prove this?

One special feature of polynomial rings is that they have many derivations (partial derivatives). With this in mind, we introduce the following idea:

Definition 4.1. A graded k-algebra R has *enough derivations* if for every non-zero homogeneous element $f \in R$ of positive degree there is a homogeneous derivation ∂ of negative degree such that $\partial(f) \neq 0$.

Example 4.2. Suppose that $R = k[y_i]_{i \in I}$ is a polynomial ring with k of characteristic 0. Then R has enough derivations: if f is a non-zero homogeneous element of positive degree then $\frac{\partial f}{\partial y_i}$ is non-zero for some i.

As the above example hints at, the concept of "enough derivations" isn't the right thing to use in positive characteristic. For the rest of the lecture, we therefore restrict to characteristic 0. (See the exercises for what happens in positive characteristic.)

Theorem 4.3. *Let R be a graded k-algebra. Then R is (isomorphic to) a polynomial ring if and only if it has enough derivations.*

Proof. It is enough to show that if R has enough derivations then it is a polynomial ring. We give a proof in the case that R is finitely generated; the general case uses the same ideas, but is technically a little more complicated (see Exercise 4.6).

Let $x_1, \ldots, x_n$ be homogeneous elements of R whose images form a k-basis of R_+/R_+^2. The x_i's generate R by Nakayama's lemma. Applying a permutation if necessary, we assume $\deg(x_1) \leq \cdots \leq \deg(x_n)$. We show that $x_1, \ldots, x_r$ are algebraically independent for $1 \leq r \leq n$ by induction on r. This will show that the map $k[T_1, \ldots, T_n] \to R$ given by $T_i \mapsto x_i$ is an isomorphism, where the T_i's are indeterminates.

Suppose that $x_1, \ldots, x_{r-1}$ are algebraically independent, but $x_1, \ldots, x_r$ are algebraically dependent. Thus there is a relation $0 = \sum_{i=0}^{d} a_i x_r^i$ with $a_i \in k[x_1, \ldots, x_{r-1}]$ such that $d > 0$ and $a_d \neq 0$. Of all such relations, choose a homogeneous one of minimal degree (by degree, we mean the common degree of each term). We will obtain a contradiction by producing a relation of smaller degree.

First suppose that a_d has positive degree. By assumption, there exists a derivation ∂ (homogeneous of negative degree) such that $\partial(a_d) \neq 0$. Applying ∂ to our relation yields a new relation of the form $0 = \partial(a_d) x_r^d + \cdots$ where the remaining terms have smaller degree in x_r. This contradicts the minimality of our initial relation. Thus a_d has degree 0, and we may as well assume $a_d = 1$.

Since the x_i are linearly independent modulo R_+^2, it follows that x_r does not belong to $k[x_1, \ldots, x_{r-1}]$. Thus $d > 1$ and $dx_r + a_{d-1}$ is non-zero. In particular, there is a derivation ∂ (homogeneous of negative degree) such that $\partial(dx_r + a_{d-1})$ is non-zero. Applying this to our original relation yields $0 = \partial(dx_r + a_{d-1})x_r^{d-1} + \cdots$ where the remaining terms have smaller degree in x_r. Note that since $\partial(x_r)$ has smaller degree than x_r, it belongs to $k[x_1, \ldots, x_{r-1}]$. This relation again contradicts the minimality of our original relation. We conclude that $x_1, \ldots, x_r$ are algebraically independent. □

Corollary 4.4. **R** *is (isomorphic to) a polynomial ring.*

Proof. It has enough derivations, just use $\frac{\partial}{\partial x_i}$ (defined on infinite series in the obvious manner). □

Remark 4.5. Corollary 4.4 is true for any coefficient field k. This lecture only proves it when k has characteristic 0.

Exercises

Exercise 4.1. Show that the ultraproduct ring **S** has enough derivations, and is therefore a polynomial ring.

Exercise 4.2. Show that the equivalence "polynomial ring if and only if enough derivations" fails in both directions in positive characteristic. [Hint: show that $k[x]/(x^p)$ has enough derivations when k has characteristic p.]

Additional exercises

Exercise 4.3. Let R be a k-algebra. A *Hasse derivation* on R is a sequence $\{\partial_i\}_{i \geq 0}$ such that each ∂_i is k-linear and $\partial_n(xy) = \sum_{i+j=n} \partial_i(x)\partial_j(y)$. The intuition is that ∂_i should be like $\frac{1}{i!}\partial_1^i$. Construct a Hasse derivation on $k[x]$ (do not assume k has characteristic 0).

Exercise 4.4. Suppose k has characteristic $p > 0$ and is perfect (every element is a pth power). We say that R has *enough Hasse derivations* if whenever f is a homogeneous element that is not a pth power there is a homogeneous Hasse derivation ∂ of negative degree such that $\partial_1(f) \neq 0$. We have the following theorem [ESS2, Theorem 2.11]: a graded k-algebra is a polynomial ring if and only if it has enough Hasse derivations. The proof is a bit involved, so we'll just look at some special cases and adjacent results:

(a) Show that a polynomial ring has enough Hasse derivations.
(b) Show that if R has enough Hasse derivations then R is reduced. (This is the first step in the proof of the theorem.)
(c) Using the theorem, show that **R** is a polynomial ring.

Exercise 4.5. Let A_n be the exterior algebra on an n-dimensional vector space, regarded as a graded algebra, and let $\mathbf{A}$ be the inverse limit of the A_n's in the category of graded algebras.

(a) Show that $\mathbf{A}$ is *not* an exterior algebra on some vector space.
(b) Presumably, $\mathbf{A}$ should be a free algebra of a certain kind; can you figure out which kind?
(c) Prove your conjecture in (b). (As far as I know, this is still an open problem, and could make a nice little paper.)

Exercise 4.6. Prove Theorem 4.3 without the assumption that R is finitely generated. (The argument needs only minor modifications.)

Notes

Theorem 4.3 originally appeared in [ESS2, §2]. See [ESS3, §4.4] for a more expository treatement. As mentioned in the exercises, [ESS2] also proves a version of Theorem 4.3 for perfect fields of positive characteristic, though the proof is more involved. A later paper [ESS4] removes the perfectness hypothesis (and the argument is yet more complicated).

Chirvasitu and Hong [CH] have extended Corollary 4.4 to non-commutative rings and Lie algebras.

LECTURE 5

Review of commutative algebra

In this lecture, we review some basic commutative algebra and also see how it extends to infinite variable polynomial rings. This is important to us since $\mathbf{R}$ and $\mathbf{S}$ are such rings.

Let R be a commutative ring. Recall that the *(Krull) dimension* of R is the supremum of n's for which there is a strict chain of prime ideals $\mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_n$. We define the *dimension* of an ideal $\mathfrak{a}$, or its vanishing locus $V(\mathfrak{a}) \subset \operatorname{Spec}(R)$, to be the Krull dimension of $R/\mathfrak{a}$.

EXAMPLE 5.1. The polynomial ring $R = k[x_1, \ldots, x_n]$ has Krull dimension n. The chain

$$(0) \subset (x_1) \subset (x_1, x_2) \subset \cdots \subset (x_1, \ldots, x_n)$$

shows that the dimension is at least n. The reverse inequality takes a little more effort to prove.

The dimension of $\mathfrak{a}$ is defined in terms of chains of primes $\mathfrak{a} \subset \mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_n$. We can also define a notion of dimension by considering chains below $\mathfrak{a}$. First suppose that $\mathfrak{a} = \mathfrak{q}$ is itself a prime ideal. The *codimension* (or *height*) of $\mathfrak{q}$ is the supremum of n's for which there exists a strict chain $\mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_n = \mathfrak{q}$; equivalently, this is the Krull dimension of the localization $R_{\mathfrak{q}}$. For a general $\mathfrak{a}$, its *codimension* is the minimum codimension of a prime $\mathfrak{q}$ containing $\mathfrak{a}$.

The above definitions behave well in "nice" rings. For example, suppose $R = k[x_1, \ldots, x_n]$. Then the dimension of $V(\mathfrak{a})$, as defined above, matches our intuition for the dimension of the algebraic set $V(\mathfrak{a}) \subset \mathbf{A}^n$. Furthermore, dimension and codimension sum to n in this case.

In general rings, (co)dimension isn't always well-behaved. However, suppose that R is a (possibly infinite) polynomial ring over a field k (or, more generally, a finitely generated k-algebra). Let $\mathfrak{a}$ be a finitely generated ideal of R. The dimension of $\mathfrak{a}$ will typically be infinite (Exercise 5.2), and so dimension is not a useful invariant. However, codimension is well-behaved in this setting. The basic reason for this is that $\mathfrak{a}$ is extended from a finite variable subring (where one has all the familiar properties of codimension), and the extension process does not change codimension:

PROPOSITION 5.2. *Let A be a finitely generated k-algebra, let R be a polynomial ring over A, and let S be a polynomial ring over R. If $\mathfrak{a}$ is a finitely generated ideal of R and $\mathfrak{b}$ is its extension to S then* $\operatorname{codim}_R(\mathfrak{a}) = \operatorname{codim}_S(\mathfrak{b})$.

COROLLARY 5.3. *Let R be as in the above proposition and let $\mathfrak{a}$ be a finitely generated ideal of R. Then* $\operatorname{codim}_R(\mathfrak{a})$ *is finite.*

Proof. Let R_0 be the A-subalgebra of R generated by the variables appearing in a finite generating set of $\mathfrak{a}$. Then $\mathfrak{a}$ is the extension of an ideal $\mathfrak{a}_0$ of R_0, and so $\operatorname{codim}_R(\mathfrak{a}) = \operatorname{codim}_{R_0}(\mathfrak{a}_0)$ be the proposition. The right side is finite by the usual (finite variable) theory. □

Here is an example of how one can use Proposition 5.2 to extend familiar results to the infinite setting.

PROPOSITION 5.4. *Let R be a polynomial ring over a finitely generated k-algebra A, and let $S = R[x]$. Let $\mathfrak{b}$ be a finitely generated ideal of S that contains a monic polynomial in x, and let $\mathfrak{a}$ be the contraction of $\mathfrak{b}$ to R. Then $\mathfrak{a}$ is finitely generated and* $\operatorname{codim}_R(\mathfrak{a}) = \operatorname{codim}_S(\mathfrak{b}) - 1$.

Proof. Choose a finite generating set for $\mathfrak{b}$, including the given monic polynomial f. Let R_0 be the A-subalgebra of R generated by all variables appearing in this generating set (other than x), and let $S_0 = R_0[x]$. Thus $\mathfrak{b}$ is the extension of an ideal $\mathfrak{b}_0$ of S_0. Let $\mathfrak{a}_0$ be the contraction of $\mathfrak{b}_0$ to R_0. One easily sees that $\mathfrak{a}$ is the extension of $\mathfrak{a}_0$, and thus finitely generated. By classical (finite variable) theory, $\operatorname{codim}_{R_0}(\mathfrak{a}_0) = \operatorname{codim}_{S_0}(\mathfrak{b}_0) - 1$; the key point here is that $R_0 \to R_0[x]/(f)$ is finite and flat. Since $\operatorname{codim}_{R_0}(\mathfrak{a}_0) = \operatorname{codim}_R(\mathfrak{a})$ and $\operatorname{codim}_{S_0}(\mathfrak{b}_0) = \operatorname{codim}_S(\mathfrak{b})$ by Proposition 5.2, the result follows. □

We now turn to regular sequences. Recall that elements $f_1, \ldots, f_r$ in a ring R form a *regular sequence* if f_i is a non-zerodivisor in the ring $R/(f_1, \ldots, f_{i-1})$ for each $1 \le i \le r$. For example, $x_1, \ldots, x_n$ is a regular sequence in $R = k[x_1, \ldots, x_n]$. The following proposition captures the most important property of regular sequences (though we will see other important properties later on):

PROPOSITION 5.5. *Let $f_1, \ldots, f_r$ be homogeneous elements of $R = k[x_1, \ldots, x_n]$. Then $f_1, \ldots, f_r$ form a regular sequence if and only if the ideal $(f_1, \ldots, f_r)$ has codimension r.*

Proof. If $f_1, \ldots, f_r$ form a regular sequence then Krull's principal ideal theorem implies that each equation $f_i = 0$ cuts down the codimenion by one, and so $(f_1, \ldots, f_r)$ has codimension r. We leave the converse as an exercise. □

REMARK 5.6. The proposition implies that if $f_1, \ldots, f_r$ is a regular sequence of homogeneous polynomials then any permutation of the sequence is still regular. This is not true (in general) for inhomogeneous polynomials.

Using Proposition 5.2, we can extend Proposition 5.5 to infinite polynomial rings:

PROPOSITION 5.7. *Let R be a (perhaps infinite) polynomial ring over k and let $f_1, \ldots, f_r \in R$. Then $f_1, \ldots, f_r$ form a regular sequence if and only if $(f_1, \ldots, f_r)$ has codimension r.*

Proof. Let R_0 be a finite variable polynomial ring containing each f_i. Then $f_1, \dots, f_r$ form a regular sequence in R_0 if and only if they form a regular sequence in R (obvious). Also, the codimension of the ideal generated by the f_i's is the same in R_0 and R by Proposition 5.2. $\square$

Exercises

Exercise 5.1. Let $\mathfrak{a} = (xz, yz) \subset k[x, y, z]$. Determine the minimal prime ideals containing $\mathfrak{a}$ and compute their codimensions.

Exercise 5.2. Let $\mathfrak{a}$ be a finitely generated ideal in an infinite polynomial ring R. Show that $\mathfrak{a}$ has infinite dimension. [This is intuitively obvious, but to actually prove it you have to exhibit long chains of prime ideals containing I.]

Exercise 5.3. Find three non-zero polynomials f_1, f_2, f_3 in $k[x, y, z]$ that do not form a regular sequence, but for which each pair does form a regular sequence.

Additional exercises

Exercise 5.4. Let $\mathfrak{p}$ be a prime of finite codimension c in a polynomial ring (over a field). Show that $\mathfrak{p}$ is finitely generated. [Hint: to do the $c = 1$ case, contract down to a finite variable polynomial ring containing a non-zero element of $\mathfrak{p}$ and then extend back. The general case can be proved by induction on c, using a similar idea.]

Exercise 5.5. Prove Proposition 5.2. First treat the case where $\mathfrak{a}$ is prime. Then do the general case; you may need to make use of Exercise 5.4 in the general case.

Notes

See [ESS2, §3] for details about dimension theory in infinite polynomial rings.

LECTURE 6

Representations of $\mathbf{GL}_n$

We work over the complex numbers in this lecture. In this lecture, and in Lectures 8 and 11, we develop the representation theory we will require when discussing **GL**-varieties. These varieties can be used to create a sort of moduli space of ideals in the Stillman regime. One can then use ideas from algebraic geometry to attack Stillman's conjecture (see Lecture 15).

Let V be a finite dimensional complex representation of the group $\mathbf{GL}_n$ of $n \times n$ invertible complex matrices. Picking a basis $v_1, \dots, v_m$ for V, the action of $\mathbf{GL}_n$ corresponds to a group homomorphism $\rho \colon \mathbf{GL}_n \to \mathbf{GL}_m$. For $g \in \mathbf{GL}_n$, let $\rho_{i,j}(g)$ be the (i,j) entry of the matrix $\rho(g)$. We say that the representation V is *rational* (resp. *polynomial*) if $\rho_{i,j}(g)$ is a rational (resp. polynomial) function of the entries of g. Of course, any polynomial representation is rational.

Here are some simple observations:

- The notion of rational or polynomial is independent of the choice of basis.
- The classes of rational and polynomial representations are closed under direct sums, tensor products, and passing to sub or quotients.
- The *standard representation* of $\mathbf{GL}_n$ is the vector space $\mathbf{C}^n$ equipped with its usual $\mathbf{GL}_n$-action. It is a polynomial representation.
- For each $k \in \mathbf{Z}$ there is a 1-dimensional representation of $\mathbf{GL}_n$ corresponding to the homomorphism $\mathbf{GL}_n \to \mathbf{GL}_1$ given by $g \mapsto (\det g)^k$. This is rational for all k and polynomial for $k \geq 0$.

The following is an extremely important fact about rational representations (see Exercise 6.8 for the proof).

THEOREM 6.1. *Every rational representation of* $\mathbf{GL}_n$ *decomposes into a direct sum of irreducible representations; i.e., rational representations are* semisimple.

Thus to understand the basic structure of rational representations, it is enough to understand the irreducible ones. For $\mathbf{GL}_1$ this is straightforward. For $k \in \mathbf{Z}$, let $\chi_k \colon \mathbf{GL}_1 \to \mathbf{GL}_1$ be the homomorphism given by $\chi_k(t) = t^k$. Then the χ_k are exactly the irreducible rational representations of $\mathbf{GL}_1$ (Exercise 6.5).

There is a mild generalization of the $\mathbf{GL}_1$ story that is very useful. Let $\mathbf{T}_n = (\mathbf{GL}_1)^n$; a group of this form is called an *algebraic torus*. A *weight* of $\mathbf{T}_n$ is an n-tuple of integers. We let $X(\mathbf{T}_n) = \mathbf{Z}^n$ be the set of all weights (the *weight lattice*). For $\lambda \in X(\mathbf{T}_n)$, let $\chi_\lambda \colon \mathbf{T}_n \to \mathbf{GL}_1$ be the homomorphism given by $\chi_\lambda(t) = t_1^{\lambda_1} \cdots t_n^{\lambda_n}$. Then once again, every rational representation of $\mathbf{T}_n$ decomposes into irreducibles, and the χ_λ account for all rational irreducible representations. (The definition of rational representation here is analogous to the $\mathbf{GL}_n$ case.)

The above result is often reformulated in the following manner. Let V be a rational representation of $\mathbf{T}_n$. Let V_λ be the set of all vectors $v \in V$ such that $t \cdot v = \chi_\lambda(t)v$ for all $t \in \mathbf{T}_n$. This is called the λ-*weight space* of V. Then V decomposes as a direct sum $\bigoplus_{\lambda \in X(\mathbf{T}_n)} V_\lambda$. We say that λ is a weight of V if $V_\lambda \neq 0$, and define its multiplicity to be $\dim(V_\lambda)$.

Now let V be a rational representation of $\mathbf{GL}_n$. Thinking of $\mathbf{T}_n$ as the group of diagonal matrices in $\mathbf{GL}_n$, we can restrict V to $\mathbf{T}_n$ and look at its weights. While $\mathbf{T}_n$ is much smaller than $\mathbf{GL}_n$, the weights of V nonetheless tell us a lot about V. To make this precise, we introduce some terminology:

- We say that a weight λ is *dominant* if $\lambda_1 \geq \cdots \geq \lambda_n$.
- We say that a weight λ is *higher* than a weight μ, written $\lambda \geq \mu$, if $\lambda - \mu$ is a $\mathbf{Z}$-linear combination of the vectors $e_i - e_j$ for $i < j$, where e_i is the ith standard basis vector of $\mathbf{Z}^n$.

We can now state a second important theorem. The proof of this theorem is a bit involved, and we simply refer to [FH, §14,15].

THEOREM 6.2. *Let V be an irreducible rational representation of $\mathbf{GL}_n$.*

(a) There exists a unique weight λ of V that is higher than all other weights; it is called the highest weight *of V.*
(b) The highest weight space V_λ is one-dimensional.
(c) The highest weight determines V up to isomorphism: if W is a second irreducible rational representation with highest weight μ then $V \cong W$ if and only if $\lambda = \mu$.
(d) The representation V is polynomial if and only if $\lambda_i \geq 0$ for all i.

Furthermore, the weights that occur as the highest weight of some irreducible rational representation are exactly the dominant weights.

Exercises

Exercise 6.1. Compute the weights of $\mathrm{Sym}^2(\mathbf{C}^3)$ as a representation of $\mathbf{GL}_3$, and draw the Hasse diagram of the dominance order.

Exercise 6.2. Compute the multiset of weights of $(\mathbf{C}^2)^{\otimes 4}$ as a representation of $\mathbf{GL}_2$. Decompose this representation into irreducible representations.

Exercise 6.3. This exercise walks you through the decomposition of the 27-dimensional representation $V = (\mathbf{C}^3)^{\otimes 3}$ of $\mathbf{GL}_3$.

(a) Compute the multiset of weights of $\bigwedge^2(\mathbf{C}^3)$, and then the multiset of weights of $W = \bigwedge^2(\mathbf{C}^3) \otimes \mathbf{C}^3$. Deduce that $W = L_{(2,1,0)} \oplus \bigwedge^3(\mathbf{C}^3)^{\oplus m}$ for some unknown multiplicity m. If you know m, how do you compute the multiset of weights of $L_{(2,1,0)}$?
(b) With what multiplicity does $\bigwedge^3(\mathbf{C}^3)$ appear as a direct summand of V? [Hint: This one-dimensional representation can only appear as a summand of a par-

ticular six-dimensional weight space. The symmetric group $S_3 \subset \mathbf{GL}_3$ acts in a particular way on this six-dimensional space, and on $\bigwedge^3(\mathbf{C}^3)$.]

(c) Use part (b) to compute m and the weights of $L_{(2,1,0)}$.

(d) Now compute the decomposition of $\mathrm{Sym}^2(\mathbf{C}^3) \otimes \mathbf{C}^3$ into irreducible representations.

(e) Now compute the decomposition of V into irreducible representations. [Hint: First decompose $\mathbf{C}^3 \otimes \mathbf{C}^3$, then tensor with $\mathbf{C}^3$.]

(f) (Come back after Lecture 8) Match your answer with the answer that Schur–Weyl duality gives.

Additional exercises

Exercise 6.4. If G acts on V, then G acts on $V^* = \{f\colon V \to \mathbf{C}\}$ by the formula $g \cdot f(v) = f(g^{-1}v)$.

(a) For $G = \mathbf{GL}_2$ and $V = \mathbf{C}^2$, write down the matrix for the action of $g \in G$ on V^*, with respect to the dual standard basis.

(b) From this you might be able to guess the general formula which takes the matrix of g on a basis of V and produces the matrix of g on the dual basis of V^*. Prove that $(\mathbf{C}^n)^* \otimes \det$ is a polynomial representation of $\mathbf{GL}_n$, whereas $(\mathbf{C}^n)^*$ is only rational (for $n > 1$).

Exercise 6.5. Prove the classification of irreducible rational representations of $\mathbf{GL}_1$.

Exercise 6.6. Let $\rho\colon \mathbf{GL}_1 \to \mathbf{GL}_2$ be a homomorphism given by

$$\rho(z) = \begin{pmatrix} z^k & r(z) \\ 0 & z^k \end{pmatrix}$$

for some $k \in \mathbf{Z}$ and some rational function $r(z)$. Deduce that $r = 0$. This does not prove the semisimplicity theorem for $\mathbf{GL}_1$, but it certainly makes it believable. [Hint: Compare certain coefficients in $\rho(z)^2 = \rho(z^2)$.]

Exercise 6.7. Let $Z \subset \mathbf{GL}_n$ be the group of scalar matrices; this is isomorphic to $\mathbf{GL}_1$. Let V be a rational representation of $\mathbf{GL}_n$ and let $V = \bigoplus_{k \in \mathbf{Z}} V_k$ be the weight decomposition for Z. Show that each V_k is a $\mathbf{GL}$-subrepresentation of V. We thus see that every rational $\mathbf{GL}$-representation admits a canonical $\mathbf{Z}$-grading.

Exercise 6.8. Let V be a rational representation of $\mathbf{GL}_n$.

(a) Let $\langle , \rangle_0$ be a Hermitian form on V. Define a new Hermitian form $\langle , \rangle$ on V by the formula

$$\langle v, w \rangle = \int_{\mathbf{U}_n} \langle gv, gw \rangle_0 \, dg.$$

Here $\mathbf{U}_n \subset \mathbf{GL}_n$ is the unitary group and dg is its Haar measure. Show that $\langle , \rangle$ is $\mathbf{GL}_n$-invariant, i.e., $\langle gv, gw \rangle = \langle v, w \rangle$ for $g \in \mathbf{GL}_n$ and $v, w \in V$. [Hint: $\mathbf{U}_n$ is Zariski dense in $\mathbf{GL}_n$.]

(b) Let W be a $\mathbf{GL}_n$-subrepresentation of V and let W' be its orthogonal complement under $\langle , \rangle$. Show that W' is also a $\mathbf{GL}_n$-subrepresentation and that $V = W \oplus W'$.

(c) Show that V decomposes as a direct sum of irreducible representations (Theorem 6.1).

Notes

The material in this lecture is standard. See [FH] for additional background. (This book works with Lie algebras, but for polynomial representations it does not make much difference.)

LECTURE 7

Regular sequences in the ultraproduct ring

Suppose that $\mathfrak{a}$ is a finitely generated homogeneous ideal in the ultraproduct ring $\mathbf{S}$. Let $f_1, \ldots, f_r$ be a homogeneous generating set, and for each $1 \le j \le r$ let $(f_{j,i})_{i \in \mathcal{I}}$ be a sequence in R_∞ that represents f_j. For $i \in \mathcal{I}$, let $\mathfrak{a}_i$ be the ideal of R_∞ generated by $f_{1,i}, \ldots, f_{r,i}$. The ideals $\mathfrak{a}_i$ are not exactly well-defined: they depend on the choice of generators for $\mathfrak{a}$ and the choice of sequences representing these generators. However, if we make different choices that yield ideals $\mathfrak{a}'_i$ then there is some set A in the ultrafilter such that $\mathfrak{a}_i = \mathfrak{a}'_i$ for all $i \in A$ (Exercise 7.1). This ambiguity will not be an issue in our applications, so we freely pass from $\mathfrak{a}$ to the $\mathfrak{a}_i$'s.

The following important theorem shows that we can pass some information between $\mathfrak{a}$ and the $\mathfrak{a}_i$'s.

THEOREM 7.1. *If $\mathfrak{a}$ is a finitely generated ideal of $\mathbf{S}$ then $\operatorname{codim}_{\mathbf{S}}(\mathfrak{a}) = \operatorname{codim}_{R_\infty}(\mathfrak{a}_i)$ for all i in a set in the ultrafilter.*

Proof. We argue as follows:

- Let $c = \operatorname{codim}_{\mathbf{S}}(\mathfrak{a})$, which is finite (Corollary 5.3). We proceed by induction on c. The $c = 0$ case is trivial. Now assume the result for $c-1$ and prove it for $c > 0$.
- Let $f = (f_i)_{i \in \mathcal{I}}$ be an element of $\mathfrak{a}$. We can find an automorphism γ_i of R_∞ such that $\gamma_i(f_i)$ is monic in x_1. The γ_i induce an automorphism γ of $\mathbf{S}$. Replacing $\mathfrak{a}$ with $\gamma(\mathfrak{a})$, we can assume that f_i is monic in x_1 for all i.
- Let $R'_\infty = k[x_2, x_3, \ldots]$ and let $\mathbf{S}'$ be their graded ultraproduct; we have $\mathbf{S} = \mathbf{S}'[x_1]$. Let $\mathfrak{a}' = \mathfrak{a} \cap \mathbf{S}'$, which is finitely generated (Proposition 5.4).
- One can show that $\mathfrak{a}'_i = R'_\infty \cap \mathfrak{a}_i$ for all i in some set in the ultrafilter. See [ESS2, Prop 4.5].
- By Proposition 5.4, we have $\operatorname{codim}_{\mathbf{S}'}(\mathfrak{a}') = \operatorname{codim}_{\mathbf{S}}(\mathfrak{a}) - 1$ and $\operatorname{codim}_{R'_\infty}(\mathfrak{a}'_i) = \operatorname{codim}_{R_\infty}(\mathfrak{a}_i) - 1$. By induction, we have $\operatorname{codim}_{\mathbf{S}'}(\mathfrak{a}') = \operatorname{codim}_{R'_\infty}(\mathfrak{a}'_i)$ for all i in some set in the ultrafilter, and so the result follows. $\square$

REMARK 7.2. It is worth emphasizing that all the dimension theory used above only works since we know that $\mathbf{S}$ is a polynomial ring!

In fact, we are most interested in the following corollary of the theorem [ESS2, Corollary 4.10]:

COROLLARY 7.3. *Let $f_1, \ldots, f_r$ be homogeneous elements of $\mathbf{S}$. Then $f_1, \ldots, f_r$ form a regular sequence in $\mathbf{S}$ if and only if $f_{1,i}, \ldots, f_{r,i}$ form a regular sequence in R_∞ for all i in some set in the ultrafilter.*

Proof. This follows from the theorem and the characterization of regular sequences via codimension in Proposition 5.7. ☐

REMARK 7.4. It is easy to see directly (without dimension theory) that if $f_{1,i}, \ldots, f_{r,i}$ forms a regular sequence for all i in some set in the ultrafilter then $f_1, \ldots, f_r$ forms a regular sequence. The converse is not so clear though. Indeed, suppose $f_1, \ldots, f_r$ is a regular sequence but $f_{1,i}, \ldots, f_{r,i}$ fails to be a regular sequence for all i; we will try to reach a contradiction.

For simplicity, say $f_{r,i}$ is a zero-divisor modulo $(f_{1,i}, \ldots, f_{r-1,i})$, and let g_i be a witness to this. One would like to argue as follows: letting g be the element of $\mathbf{S}$ defined by the g_i's, we see that gf_r is zero modulo $(f_1, \ldots, f_{r-1})$, contradicting our original assumption. The problem with this is that if the g_i's have unbounded degrees then they do not define an element of $\mathbf{S}$. Thus to make this direct argument work, one has to somehow control the degree of g_i, which seems difficult. This is why we have used codimension instead.

Exercises

Exercise 7.1. Let $f_{1,\bullet}, \ldots, f_{r,\bullet}$ and $g_{1,\bullet}, \ldots, g_{s,\bullet}$ be sequences of homogeneous elements of R_∞ indexed by I, and let $f_1, \ldots, f_r$ and $g_1, \ldots, g_s$ be the corresponding elements of $\mathbf{S}$. Define

$$\mathfrak{a}_i = (f_{1,i}, \ldots, f_{r,i}) \qquad \mathfrak{b}_i = (g_{1,i}, \ldots, g_{s,i})$$
$$\mathfrak{a} = (f_1, \ldots, f_r) \qquad \mathfrak{b} = (g_1, \ldots, g_s).$$

Show that $\mathfrak{a} = \mathfrak{b}$ if and only if $\mathfrak{a}_i = \mathfrak{b}_i$ for all i belonging to some set in the ultrafilter.

Exercise 7.2. Let $\mathfrak{a}$ be a finitely generated ideal of $\mathbf{S}$ and let $\mathfrak{a}_\bullet$ be the corresponding sequence of ideals in R_∞. Show that $\mathfrak{a}$ is radical if and only if $\mathfrak{a}_i$ is radical for all i in some set in the ultrafilter. To do this, make use of the following result:

($*$) Given degrees $d_1, \ldots, d_r$ there exists a positive integer $M = M(d_1, \ldots, d_r)$ with the following property: if $\mathfrak{a}$ is an ideal in a polynomial ring generated by r homogeneous elements of degrees $d_1, \ldots, d_r$ then $\mathrm{rad}(\mathfrak{a})^M \subset \mathfrak{a}$.

This statement is a consequence of the *effective Nullstellensatz*.

Exercise 7.3. Prove a version of Theorem 7.1 for the inverse limit ring $\mathbf{R}$.

Notes

Theorem 7.1 was originally proved in [ESS2, Theorem 4.8].

LECTURE 8

Polynomial functors

We work over the complex numbers in this lecture. Two important examples of polynomial representations of $\mathbf{GL}_n$ are the symmetric powers $\mathrm{Sym}^k(\mathbf{C}^n)$ and exterior powers $\bigwedge^k(\mathbf{C}^n)$. We have isomorphisms

$$\mathrm{Sym}^k(\mathbf{C}^n) \cong \mathrm{Hom}_{\mathfrak{S}_k}(\mathrm{triv}, (\mathbf{C}^n)^{\otimes k}), \qquad \textstyle\bigwedge^k(\mathbf{C}^n) \cong \mathrm{Hom}_{\mathfrak{S}_k}(\mathrm{sgn}, (\mathbf{C}^n)^{\otimes k}),$$

where $\mathfrak{S}_k$ denotes the symmetric group, and triv and sgn are its trivial and sign representations. The action of $\mathfrak{S}_k$ on $(\mathbf{C}^n)^{\otimes k}$ is by permuting tensor factors. We thus see that, in a sense, Sym^k corresponds to the trivial representation of $\mathfrak{S}_k$ and $\bigwedge^k$ to the sign representation. Do the other representations of $\mathfrak{S}_k$ play a role here? Yes!

Recall that a *partition* is a sequence $\lambda = (\lambda_1, \lambda_2, \ldots)$ of non-negative integers that is non-increasing ($\lambda_i \geq \lambda_{i+1}$) and eventually 0 ($\lambda_i = 0$ for $i \gg 0$). The *size* of λ is $|\lambda| = \sum_{i \geq 1} \lambda_i$, and the *length* of λ, denoted $\ell(\lambda)$, is the number of non-zero parts (i.e., the maximum r such that $\lambda_r \neq 0$). We say that λ is a partition of $|\lambda|$.

Conjugacy classes in $\mathfrak{S}_k$ are naturally in bijection with partitions of k, via cycle decompositions. By finite group representation theory, it follows that the number of complex irreducible representations of $\mathfrak{S}_k$ is the number of partitions of k. In fact, there is a natural bijection here: for a partition λ of k, there is an associated representation $\mathbf{Sp}_\lambda$ of $\mathfrak{S}_k$ called the *Specht module*. Two simple examples:

$$\mathbf{Sp}_{(k)} = \mathrm{triv}, \qquad \mathbf{Sp}_{(1^k)} = \mathrm{sgn}\,.$$

A more interesting example: $\mathbf{Sp}_{(k-1,1)}$ is the standard representation, i.e., the kernel of the augmentation map $\mathbf{C}^k \to \mathbf{C}$.

Now, for a vector space V and a partition λ of k, put

$$\mathbf{S}_\lambda(V) = \mathrm{Hom}_{\mathfrak{S}_k}(\mathbf{Sp}_\lambda, V^{\otimes k}).$$

Thus

$$\mathbf{S}_{(k)}(V) = \mathrm{Sym}^k(V), \qquad \mathbf{S}_{(1^k)}(V) = \textstyle\bigwedge^k(V).$$

The construction $\mathbf{S}_\lambda(V)$ is functorial in V: if $V \to W$ is a linear map then there is an induced linear map $\mathbf{S}_\lambda(V) \to \mathbf{S}_\lambda(W)$. In particular, $\mathbf{GL}(V)$ acts on $\mathbf{S}_\lambda(V)$, and so $\mathbf{S}_\lambda(\mathbf{C}^n)$ is a representation of $\mathbf{GL}_n$. The functor $\mathbf{S}_\lambda$ is called the *Schur functor* associated to λ. A *polynomial functor* is a direct sum of Schur functors, or a functor isomorphic to such. We allow infinite direct sums, since we want the symmetric algebra $\mathrm{Sym} = \bigoplus_{k \geq 0} \mathrm{Sym}^k$ to be a polynomial functor.

The following key result connects Schur functors and highest weight theory, and demonstrates an intimate connection between the representation theories of symmetric groups and general linear groups.

THEOREM 8.1. *Let λ be a partition and let n be a non-negative integer.*

(a) *If $n < \ell(\lambda)$ then $\mathbf{S}_\lambda(\mathbf{C}^n) = 0$.*
(b) *If $n \geq \ell(\lambda)$ then $\mathbf{S}_\lambda(\mathbf{C}^n)$ is the irreducible polynomial representation of $\mathbf{GL}_n$ with highest weight λ (or really, $(\lambda_1, \ldots, \lambda_n)$).*

Proof. See [FH, §6, §15.5]. $\square$

The above theorem shows that Schur functors can be used to construct the irreducible representation of $\mathbf{GL}_n$ with highest weight λ in a uniform manner as n varies. This is the main reason that Schur functors are interesting from the perspective of representation theory. However, our interest in polynomial functors stems from their appearance in algebraic geometry. We illustrate this with an example.

Let X_n be the space of symmetric $n \times n$ matrices, regarded as an algebraic variety. Let $Y_{n,r} \subset X_n$ be the subset consisting of matrices of rank at most r. This is a closed subvariety: it is defined by the vanishing of principal minors of size $r+1$. The $Y_{n,r}$ are examples of *determinantal varieties*, which have been intensely studied in algebraic geometry, commutative algebra, and representation theory.

Let us reformulate the above example in a coordinate-free manner. An $n \times n$ symmetric matrix defines a symmetric bilinear form on $\mathbf{C}^n$, i.e., a linear map $\mathrm{Sym}^2(\mathbf{C}^n) \to \mathbf{C}$. We can therefore identify X_n with $\mathrm{Sym}^2(\mathbf{C}^n)^*$. For a finite dimensional vector space V, let $X(V) = \mathrm{Sym}^2(V)^*$; this specializes to X_n with $V = \mathbf{C}^n$. Let $Y_r(V) \subset X(V)$ be the rank $\leq r$ subvariety.

For any finite dimensional vector space W, we can regard W^* as a variety, and its coordinate ring is the symmetric algebra $\mathrm{Sym}(W)$ (which is a polynomial ring in the basis vectors of W). Thus the coordinate ring $R(V)$ of $X(V)$ is given by

$$R(V) = \mathrm{Sym}(\mathrm{Sym}^2(V)).$$

This is a polynomial functor (see Exercise 8.1). Let $\mathfrak{a}_r(V) \subset R(V)$ be the radical ideal defining $Y_r(V)$. Using the description of $\mathfrak{a}_r(V)$ in terms of principal minors, it is not difficult to see that $\mathfrak{a}_r$ is a subfunctor of R (and thus itself a polynomial functor).

We thus see that the coordinate rings of the $Y_{n,r}$'s can be described uniformly (in n) by a single polynomial functor. This can be a very useful persepctive, especially if one is interested in properties of $Y_{n,r}$ that are independent of n.

Exercises

Exercise 8.1. Put $T_k(V) = V^{\otimes k}$. Show that the following conditions on a functor F are equivalent:

(a) F is polynomial.
(b) F is a subfunctor of a direct sum of T_k's.
(c) F is a direct summand of a direct sum of T_k's.

As an application, show that a tensor product or composition of polynomial functors is a polynomial functor.

Exercise 8.2. Let V be a vector space. Show that $\mathbf{S}_{(n-1,1)}(V)$ is naturally isomorphic to the kernel of the multiplication map $\mathrm{Sym}^{n-1}(V) \otimes V \to \mathrm{Sym}^n(V)$.

Exercise 8.3. Let F be a polynomial functor. Define $\ell(F)$ to be the supremum of the set
$$\{\ell(\lambda) \mid \mathbf{S}_\lambda \text{ is a summand of } F\}$$
Suppose that $\ell(F) = n$ is finite (we say that F is *bounded*). Show that the function
$$\{\text{subfunctors of } F\} \to \{\mathbf{GL}_n\text{-subrepresentations of } F(\mathbf{C}^n)\}$$
$$G \mapsto G(\mathbf{C}^n)$$
is a well-defined bijection. (This gives a precise sense in which evaluating on $\mathbf{C}^n$ does not lose information.)

Additional exercises

Exercise 8.4. Let V and W be vector spaces. Show that there are natural isomorphisms
$$\mathrm{Sym}^n(V \oplus W) = \bigoplus_{i+j=n} \mathrm{Sym}^i(V) \otimes \mathrm{Sym}^j(W)$$
$$\mathrm{Sym}^n(V \otimes W) = \bigoplus_{\lambda \vdash n} \mathbf{S}_\lambda(V) \otimes \mathbf{S}_\lambda(W)$$
These are known as the *binomial theorem* and *Cauchy identity*. [Hint: for the Cauchy identity, decompose $V^{\otimes n}$ and $W^{\otimes n}$ using Schur–Weyl duality, then tensor these together and take $\mathfrak{S}_n$-invariants. Second hint: irreducible representations of $\mathfrak{S}_n$ are self-dual.]

Exercise 8.5. In what follows, V denotes a vector space.

(a) Let $S(V) = \mathrm{Sym}(\mathbf{C}^2 \otimes V)$. Show that S is a polynomial functor.
(b) Show that S is bounded, in the sense of Exercise 8.3.
(c) An *ideal* of S is a subfunctor $\mathfrak{a}$ such that $\mathfrak{a}(V)$ is an ideal of $S(V)$ for all V. Show that ideals of S satisfy the ascending chain condition. [Hint: use Exercise 8.3.]

Exercise 8.6. Look at the six-dimensional $(1, 1, 1)$ weight space inside $(\mathbf{C}^3)^{\otimes 3}$. This has an action of $\mathfrak{S}_3$ by permuting the tensor factors, and a separate action of $\mathfrak{S}_3 \subset \mathbf{GL}_3$. Using either action to identify this weight space with the regular representation of $\mathfrak{S}_3$ on $\mathbf{C}[\mathfrak{S}_3]$, identify the other action.

Exercise 8.7. Consider the Specht representation of $\mathfrak{S}_4$ for the partition $\lambda = (2, 2)$, living inside the polynomial ring $\mathbf{C}[x_1, x_2, x_3, x_4]$.

(a) Identify the basis of polynomials associated to standard Young tableaux.
(b) There's a non-standard tableau whose polynomial is $(x_1-x_4)(x_2-x_3)$. Express this polynomial in terms of the basis. You may also want to practice by expressing the polynomials associated to other non-standard tableaux in terms of the basis.
(c) Prove that this Specht representation is irreducible.
(d) Let $s = (12) \in \mathfrak{S}_4$. Find an eigenbasis for s. Show that this is a simultaneous eigenbasis for the four *Young-Jucys-Murphy operators*

$$j_1 = 0, \qquad j_2 = (12), \qquad j_3 = (13) + (23), \qquad j_4 = (14) + (24) + (34)$$

in $\mathbf{C}[\mathfrak{S}_4]$. Find the eigenvalues of these YJM operators, and try to relate them to the standard young tableaux of shape λ.
(e) The Specht representation generates an $\mathfrak{S}_4$-invariant ideal $I_\lambda \subset \mathbf{C}[x_1, x_2, x_3, x_4]$, whose vanishing set is an $\mathfrak{S}_4$-invariant algebraic set inside $\mathbf{C}^4$. Describe this vanishing set.

Exercise 8.8. Compute all irreducible representations of $\mathfrak{S}_4$ and their dimensions. For $n \in \{2, 3, 4\}$, write down the decomposition of $(\mathbf{C}^n)^{\otimes 4}$ given by Schur–Weyl duality. Compute the dimensions of L_λ for each summand in this decomposition, and make sure the overall dimensions add up appropriately.

Exercise 8.9. Let V be a rational $\mathbf{GL}_n$ representation. Prove that $\mathrm{End}_{\mathbf{GL}_n}(V)$ is semisimple. Prove that the multiplicity spaces in V are irreducible modules for $\mathrm{End}_{\mathbf{GL}_n}(V)$. [Hint: What are the irreducible representations of a matrix algebra? Of a product of matrix algebras?]

Notes

The material in this lecture is standard. Symmetric group representations are discussed in [FH, §4]. Schur functors and polynomial functors are discuss in [FH, §6] and [Ma, I.A]. All this material is also discussed at length in [SS]. The notation for Specht modules is usually something like $\mathbf{S}^\lambda$; we used a different notation to try and avoid confusion with Schur functors.

LECTURE 9

Small subalgebras

Recall that Principle 1.7 states that polynomials of high collective strength behave approximately like independent variables. As independent variables forms a regular sequence, this suggests that polynomials of high collective strength should form a regular sequence. We now prove this:

THEOREM 9.1. *Given integers $d_1, \ldots, d_r$ there is an integer $B = B(d_1, \ldots, d_r)$ with the with the following property. If $f_1, \ldots, f_r \in k[x_1, \ldots, x_n]$ are homogeneous of degrees $d_1, \ldots, d_r$ and collective strength $> B$ then $f_1, \ldots, f_r$ form a regular sequence.*

We emphasize that the key aspect of the theorem is that the quantity B is independent of the number of variables n.

Proof. We argue as follows:

- Suppose the statement is false. Then for each positive integer i we can find a collection $(f_{1,i}, \ldots, f_{r,i})$ of polynomials (in some number of variables) of strength at least i which does not form a regular sequence. We treat the f's as elements of R_∞.
- Let $f_1, \ldots, f_r$ be the elements of $\mathbf{S}$ defined by $f_{1,\bullet}, \ldots, f_{r,\bullet}$. As we have seen (Exercise 3.2), $f_1, \ldots, f_r$ have infinite collective strength. (Here we take the index set $\mathcal{I}$ for the ultraproduct to be the positive integers.)
- Since $\mathbf{S}$ is a polynomial ring (Exercise 3.2), it follows that $f_1, \ldots, f_r$ forms a regular sequence in $\mathbf{S}$; in fact, the f_i's can be taken to be "variables" in the isomorphism of $\mathbf{S}$ with a polynomial ring.
- By Corollary 7.3, we see that $f_{1,i}, \ldots, f_{r,i}$ forms a regular sequence for all i in some set in the ultrafilter; this is a contradiction. $\square$

We now come to a very important result: the existence of small subalgebras. We begin with an informal discussion. Let $f_1, \ldots, f_r$ be homogeneous polynomials. If $f_1, \ldots, f_r$ have "large" collective strength then they form a regular sequence by the above theorem. Otherwise, some homogeneous linear combination can be expressed using a "small" number of elements of lower degrees. We can then apply this same observation to the elements appearing in this expression. Continuing in this manner, we see that $f_1, \ldots, f_r$ can be expressed in terms of elements $g_1, \ldots, g_s$ which have "large" collective strength and with s "small." The above theorem implies that $g_1, \ldots, g_s$ form a regular sequence. Thus, in any case, $f_1, \ldots, f_r$ belong to a subalgebra generated by a regular sequence of "small" length. This is the small subalgebra.

We now formalize the above discussion:

Theorem 9.2. *Fix postive integers $d_1, \ldots, d_r$. Then there exists an integer $C = C(d_1, \ldots, d_r)$ with the following property. If $f_1, \ldots, f_r$ are homogeneous elements of $k[x_1, \ldots, x_n]$ of degrees $d_1, \ldots, d_r$ then there exists a homogeneous regular sequence $g_1, \ldots, g_s$ with $\deg(g_j) \le \max(d_1, \ldots, d_r)$ and $s \le C$ such that each f_i belongs to the subalgebra $k[g_1, \ldots, g_s]$.*

Again, the key point is that C is independent of the number of variables n.

Proof. Let $B = B(d_1, \ldots, d_r)$ be as in Theorem 9.1. If the collective strength of $f_1, \ldots, f_r$ is $> B$ then the f_i's form a regular sequence by Theorem 9.1, and so we can take $(g_1, \ldots, g_s) = (f_1, \ldots, f_r)$.

Now suppose that $(f_1, \ldots, f_r)$ have collective strength $\le B$. Making a linear change of variables, we may as well assume we have an expression $f_r = \sum_{i=1}^{B} a_i b_i$ where the a_i and b_i are homogeneous of degree $< d_r$. Let $d'_i = \deg(a_i)$ and $d''_i = \deg(b_i)$. The tuple

$$\mathbf{e} = (d_1, \ldots, d_{r-1}, d'_1, \ldots, d'_B, d''_1, \ldots, d''_B)$$

is smaller than the tuple $\mathbf{d} = (d_1, \ldots, d_r)$, in the sense that d_r has been replaced by a list of numbers that are all strictly smaller than it. By induction, we can therefore assume that result holds in degree $\mathbf{e}$. (One should think about why induction is justified here: the key point is that our degree tuples are well-ordered.) Thus there is a homogeneous regular sequence $g_1, \ldots, g_s$ with $\deg(g_j) \le \max(\mathbf{e}) \le \max(\mathbf{d})$ and $s \le C(\mathbf{e})$ such that $f_1, \ldots, f_{r-1}$ and the a's and b's belong to $k[g_1, \ldots, g_s]$. Of course, this implies that f_r also belongs to $k[g_1, \ldots, g_s]$. ☐

Remark 9.3. The proof shows that we can take $C(\mathbf{d})$ to be the maximum of $B(\mathbf{d})$ and $C(\mathbf{e})$ as $\mathbf{e}$ varies over all ways of replacing one d_i with a list of at most $2B(\mathbf{d})$ strictly smaller numbers (really $B(\mathbf{d})$ pairs of positive integers summing to d_i).

Exercises

Exercise 9.1. Show that if f_1 and f_2 are linearly independent irreducible homogeneous polynomials then f_1, f_2 is a regular sequence. Conclude that we can take $B(d_1, d_2) = 2$ in Theorem 9.1.

Exercise 9.2. A priori, the bound $B(d_1, \ldots, d_r)$ also depends on the field k. Show that it is in fact independent of k, at least if we work in characteristic 0. [Hint: consider the graded ultraproduct of rings $k_i[x_1, x_2, \ldots]$ for varying fields k_i. Show that this is a polynomial ring, and then carry out our same arguments.]

Notes

Theorem 9.1 was originally proved by Ananyan–Hochster [AH, Theorems A]. The proof given here is different from the original one, and comes from [ESS2, Theorem 4.11]. The existence of small subalgebras (Theorem 9.2) was also originally by

Ananyan–Hochster [AH, Theorems B]. The proof given here is essentially the same as the one given there, and in [ESS2, Theorem 4.12]. For more expository treatments, see [ESS3, §6] (for regular sequences) and [ESS3, §8] (for small subalgebras).

Some explicit results are known for the quantity B in Theorem 9.1; see [Ch].

Lecture 10

Stillman's conjecture via ultraproducts

Stillman's conjecture follows from the existence of small subalgebras (Theorem 9.2) and standard arguments in commutative algebra. In this lecture, we explain these standard arguments and then complete the proof of Stillman's conjecture.

Suppose that R is a ring, x_1 and x_2 are elements of R, and we want to find the projective resolution of $R/(x_1, x_2)$. The first few terms are easy:

$$Re_1 \oplus Re_2 \to R \to R/(x_1, x_2) \to 0$$

where the first map takes e_i to x_i. There is one obvious element in the kernel, namely $x_2e_1 - x_1e_2$. It is thus reasonable to consider the complex

$$\cdots \to 0 \to R \to Re_1 \oplus Re_2 \to R \to R/(x_1, x_2) \to 0.$$

This is not necessarily exact. However, it is the only complex we can really write down without knowing more information about the x's. This "universal" complex is the (augmented) Koszul complex on x_1 and x_2.

We now explain the general construction. Let $x_1, \ldots, x_n$ be elements of R. Let $K_r = \bigwedge^r(R^n)$ and define a differential $d\colon K_r \to K_{r-1}$ by

$$d(e_{\alpha_1} \wedge \cdots \wedge e_{\alpha_r}) = \sum_{i=1}^{r} (-1)^{i+1} x_{\alpha_1} \cdot e_{\alpha_1} \wedge \cdots \wedge \hat{e}_{\alpha_i} \wedge \cdots \wedge e_{\alpha_r}.$$

One readily verifies that $d^2 = 0$ and so $K_\bullet$ is a complex. (Note that $R/(x_1, \ldots, x_n)$ is not a term in this complex.) This is the *Koszul complex* on $x_1, \ldots, x_n$. We write $K(x_1, \ldots, x_n)$ to indicate the dependence on the x's, when needed. The homology of this complex is called *Koszul homology.* Note that $\mathrm{H}_0(K) = R/(x_1, \ldots, x_n)$.

Proposition 10.1. *Let $K' = K(x_1, \ldots, x_{n-1})$ and $K = K(x_1, \ldots, x_n)$. Then we have an exact sequence of complexes*

$$0 \to K' \to K \to K'[-1] \to 0$$

and a long exact sequence in Koszul homology

$$\cdots \longrightarrow \mathrm{H}_i(K') \xrightarrow{x_n} \mathrm{H}_i(K') \longrightarrow \mathrm{H}_i(K) \longrightarrow \mathrm{H}_{i-1}(K') \xrightarrow{x_n} \mathrm{H}_{i-1}(K') \longrightarrow \cdots$$

Proof. We have an obvious inclusion $K' \to K$. The map $K \to K'[-1]$ in degree i is the map $\bigwedge^i(R^n) \to \bigwedge^{i-1}(R^{n-1})$ that takes $e_{\alpha_1} \wedge \cdots \wedge e_{\alpha_{n-1}} \wedge e_n$ to $e_{\alpha_1} \wedge \cdots \wedge e_{\alpha_{n-1}}$, and kills other basis vectors. One easily checks that these are maps of complexes, and it is clear that we have a short exact sequence. The description of the connecting homomorphism in the long exact sequence comes from simply tracing the definitions. □

Corollary 10.2. *Suppose that* $x_1, \dots, x_n$ *is a regular sequence in* R. *Then* $K = K(x_1, \dots, x_n)$ *is exact in positive degrees. In particular, the augmented complex* $K \to R/(x_1, \dots, x_n)$ *is a projective resolution.*

Proof. Let K' be as above. By induction on n, we see that K' is exact in positive degrees. The long exact sequence above shows that K is exact in degrees ≥ 2. In degree 1, we see that $\mathrm{H}_1(K)$ is the kernel of multiplication by x_n on $\mathrm{H}_0(K') = R/(x_1, \dots, x_{n-1})$. Since we have a regular sequence, this map is injective. $\square$

The following example is extremely useful:

Example 10.3. Let $R = k[x_1, \dots, x_n]$ be a polynomial ring. Then $x_1, \dots, x_n$ is a regular sequence, and $R/(x_1, \dots, x_n) = k$. The Koszul complex gives a projective resolution $\bigwedge^\bullet(R^n) \to k$.

The following is the key result we need, and why regular sequences are so important in our small subalgebras:

Proposition 10.4. *Let* R *be a graded* k*-algebra, let* $g_1, \dots, g_m$ *be a homogeneous regular sequence in* R, *and let* $S = k[t_1, \dots, t_m]$. *Consider the* k*-algebra homomorphism* $S \to R$ *given by* $t_i \mapsto g_i$. *Then* R *is flat over* S. *In particular, if* $\mathfrak{a}$ *is an ideal of* S *then* $\mathrm{pdim}_R(R/\mathfrak{a}^e) \leq \mathrm{pdim}_S(S/\mathfrak{a})$.

Proof. By the above example, we have a projective resolution $K(t_1, \dots, t_m) \to k$ given by the Koszul complex. We thus see that $\mathrm{Tor}^S_\bullet(R, k)$ is computed by the complex $K(t_1, \dots, t_m) \otimes_S R$, which is clearly isomorphic to $K(g_1, \dots, g_m)$. Since $g_1, \dots, g_m$ is a regular sequence, this is exact in positive degrees, and so $\mathrm{Tor}^S_i(R, k) = 0$ for $i > 0$. This implies that R is flat over S.

Now let $\mathfrak{a}$ be an ideal of S. Let $P_\bullet \to S/\mathfrak{a}$ be a projective resolution over S of length $d = \mathrm{pdim}_S(S/\mathfrak{a})$. Then $R \otimes_S P_\bullet \to R \otimes S/\mathfrak{a} = R/\mathfrak{a}^e$ is a projective resolution over R of length d. $\square$

We can now prove Stillman's conjecture.

Theorem 10.5. *Given positive integers* $d_1, \dots, d_r$, *there exists* $N = N(d_1, \dots, d_r)$ *such that if* $\mathfrak{a}$ *is an ideal of* $R = k[x_1, \dots, x_n]$ *generated by* r *elements of degrees* $d_1, \dots, d_r$ *then* $\mathrm{pdim}_R(R/\mathfrak{a}) \leq N$.

Proof. Let $f_1, \dots, f_r$ in $k[x_1, \dots, x_n]$ of degrees $d_1, \dots, d_r$ be given. By existence of small subalgebras (Theorem 9.2). the f_i's belong to some $S = k[g_1, \dots, g_s]$ where the g_i's are a regular sequence and $s \leq C = C(d_1, \dots, d_r)$; note that $S \cong k[t_1, \dots, t_s]$. Let $\mathfrak{a}_0$ be the ideal of $k[g_1, \dots, g_s]$ generated by the f_i's, so that $\mathfrak{a}$ is the extension of $\mathfrak{a}_0$. By the previous proposition, $\mathrm{pdim}_R(R/\mathfrak{a}) \leq \mathrm{pdim}_S(S/\mathfrak{a}_0)$, and the latter is at most s by Hilbert's syzygy theorem. We can thus take $N = C$. $\square$

Exercises

Exercise 10.1. We have seen that if $f_1, \dots, f_r$ is a regular sequence then $f_1, \dots, f_r$ is algebraically independent. Given an example of $f_1, \dots, f_r \in k[x_1, \dots, x_n]$ that are homogeneous of positive degree and algebraically independent, but not a regular sequence.

Exercise 10.2. Let R be a graded ring (in non-negative degrees) with $R_0 = k$ a field. Let $f_1, \dots, f_r$ be homogeneous elements of positive degree such that we have $\mathrm{H}_1(K(f_1, \dots, f_r)) = 0$. Show that $f_1, \dots, f_r$ is a regular sequence.

Exercise 10.3. Describe the top Koszul homology group $\mathrm{H}_n(K(x_1, \dots, x_n))$, for arbitrary elements $x_1, \dots, x_n$ in a ring R.

Notes

See [ESS3, §8] for additional details.

LECTURE 11

Representations of $\mathbf{GL}_\infty$

We work over the complex numbers in the remaining lectures. Let F be a polynomial functor. We can "probe" F by evaluating it on $\mathbf{C}^n$. This might give us zero, or something non-zero but still degenerate, if n is too small. If F is a finite sum of Schur functors, the above theorem essentially tells us that if n is sufficiently large, we can see "enough" by evaluating on $\mathbf{C}^n$. However, if F is an infinite sum of Schur functors, we might never see the full picture by evaluating on $\mathbf{C}^n$.

EXAMPLE 11.1. Let $F = \bigoplus_{m \geq 0} \bigwedge^m$. This is an infinite sum of Schur functors. However,

$$F(\mathbf{C}^n) = \bigoplus_{m=0}^{n} \bigwedge^m(\mathbf{C}^n)$$

is a finite sum of irreducible representations of $\mathbf{GL}_n$. Thus we never see all of F by evaluating on $\mathbf{C}^n$.

EXAMPLE 11.2. Recall from Lecture 8 that $R(V) = \mathrm{Sym}(\mathrm{Sym}^2(V))$ is the coordinate ring of the space of symmetric bilinear forms on V, and $\mathfrak{a}_r(V) \subset R(V)$ is the ideal defining the rank $\leq r$ locus. Every symmetric bilinear form on $\mathbf{C}^n$ has rank at most n, and so $\mathfrak{a}_r(\mathbf{C}^n) = 0$ for $n \geq r$. This shows that we cannot distinguish the various $\mathfrak{a}_r$'s (as subfunctors of R) by evaluating on a single $\mathbf{C}^n$.

This above discussion suggests it might be helpful to evaluate polynomial functors on infinite dimensional spaces. We now explain how this works.

Let $\mathbf{V} = \bigcup_{n \geq 1} \mathbf{C}^n$ and let $\mathbf{GL} = \bigcup_{n \geq 1} \mathbf{GL}_n$. Note that $\mathbf{V}$ is naturally a representation of $\mathbf{GL}$; we call it the *standard representation*. Using the fact that $\mathbf{S}_{\underline{\lambda}}(\mathbf{C}^n)$ is an irreducible of $\mathbf{GL}_n$ for large n, it is not hard to see that $\mathbf{S}_{\underline{\lambda}}(\mathbf{V})$ is an irreducible representation of $\mathbf{GL}$. We say that a representation of $\mathbf{GL}$ is *polynomial* if it is isomorphic to a (perhaps infinite) direct sum of representations of the form $\mathbf{S}_{\underline{\lambda}}(\mathbf{V})$. The category $\mathrm{Rep}^{\mathrm{pol}}(\mathbf{GL})$ of polynomial representations is a semi-simple abelian category, with the simple objects being the $\mathbf{S}_{\underline{\lambda}}(\mathbf{V})$, and is closed under tensor product.

It is not difficult to deduce from the above discussion that the functor

$$\begin{aligned} \{\text{polynomial functors}\} &\to \mathrm{Rep}^{\mathrm{pol}}(\mathbf{GL}) \\ F &\mapsto F(\mathbf{V}) \end{aligned}$$

is an equivalence of categories. In other words, evaluating a polynomial functor on $\mathbf{V}$ does not lose information (assuming we keep track of the resulting $\mathbf{GL}$ action).

The previous paragraph shows that one can pass back and forth from polynomial functors to polynomial representations. Polynomial representations have some advantages over polynomial functors. For one, they are more concrete: there is just one vector space and one group action to keep track of. Another advantage is that one can apply ideas from the representation theory of $\mathbf{GL}_n$ to polynomial representations. In particular, one has a theory of weights, which we now explain.

Let $\mathbf{T}$ be the subgroup of $\mathbf{GL}$ consisting of diagonal matrices. A *weight* of $\mathbf{T}$ is a tuple $\lambda = (\lambda_1, \lambda_2, \ldots)$ of integer such that $\lambda_i = 0$ for $i \gg 0$. Given λ, we let $\chi_\lambda \colon \mathbf{T} \to \mathbf{C}^\times$ be the homomorphism defined by

$$\chi_\lambda(t) = t_1^{\lambda_1} t_2^{\lambda_2} \cdots .$$

Here t is a diagonal matrix and t_1, t_2, and so on are its diagonal entries. Suppose that V is a polynomial representation. We define the λ-weight space V_λ to be the set of vectors $v \in V$ such that $t \cdot v = \chi_\lambda(t) \cdot v$ for all $t \in \mathbf{T}$. One can show that V is the direct sum of its weight spaces V_λ as λ varies.

Exercises

Exercise 11.1. Let V be a finite length polynomial representation of $\mathbf{GL}$. Show that the infinite symmetric group $\mathfrak{S}$ acts with finitely many orbits on the set of weights appearing in V.

Exercise 11.2. Let V be a polynomial representation of $\mathbf{GL}$ and let V_{1^k} denote its 1^k weight space under $\mathbf{T}$. (1^k means the weight $(1, \ldots, 1, 0, 0, \ldots)$, where there are k 1's.)

(a) Show that V_{1^k} is naturally a representation of the symmetric group $\mathfrak{S}_k$.
(b) When $V = \mathbf{S}_\lambda(\mathbf{V})$ show that V_{1^k} is the Specht module $\mathbf{Sp}_\lambda$.

Additional exercises

Exercise 11.3. Let $\mathrm{Rep}(\mathfrak{S}_*)$ be the category whose objects are sequences $(M_n)_{n \ge 0}$ where M_n is a complex representation of the symmetric group $\mathfrak{S}_n$.

(a) Show that there is a unique (up to isomorphism) equivalence of categories

$$\Phi \colon \mathrm{Rep}(\mathfrak{S}_*) \to \mathrm{Rep}^{\mathrm{pol}}(\mathbf{GL})$$

satisfying $\Phi(\mathbf{Sp}_\lambda) = \mathbf{S}_\lambda(\mathbf{V})$. (Here we regard $\mathbf{Sp}_\lambda$ as the sequence that is only non-zero in index $|\lambda|$.)

(b) Via Φ, we can transport the tensor product on $\mathrm{Rep}^{\mathrm{pol}}(\mathbf{GL})$ to $\mathrm{Rep}(\mathfrak{S}_*)$. What do we get?

Notes

See [SS] for more background on polynomial representations.

Lecture 12

GL-varieties

A **GL**-*algebra* is a **C**-algebra equipped with an action of **GL** by algebra automorphisms under which it forms a polynomial representation. A simple example is $\mathrm{Sym}(V)$, where V is a polynomial representation of **GL**. We say that a **GL**-algebra R is *finitely* **GL**-*generated* if it is generated as a k-algebra by the **GL**-orbits of finitely many elements; equivalently, R is a quotient of $\mathrm{Sym}(V)$ for some finite length polynomial representation V. We now come to an important definition:

Definition 12.1. An *affine* **GL**-*variety* is a scheme of the form $\mathrm{Spec}(R)$ where R is a reduced and finitely **GL**-generated **GL**-algebra.

Note that **GL**-varieties are almost always infinite dimensional. The following definition introduces the most important **GL**-varieties:

Definition 12.2. For a partition λ, define $\mathbf{A}^{\lambda}$ to be the spectrum of the **GL**-algebra $\mathrm{Sym}(\mathbf{S}_{\lambda})$. For a tuple $\underline{\lambda} = [\lambda_1, \ldots, \lambda_n]$ of partitions, put $\mathbf{A}^{\underline{\lambda}} = \mathbf{A}^{\lambda_1} \times \cdots \times \mathbf{A}^{\lambda_n}$.

These **GL**-varieties play the role of the familiar $\mathbf{A}^n$'s in classical algebraic geometry: every affine **GL**-variety is a closed **GL**-subvariety of some $\mathbf{A}^{\underline{\lambda}}$. Let's look at some special cases now.

Example 12.3. Consider the case $\lambda = (d)$, so that $\mathbf{S}_{\lambda} = \mathrm{Sym}^d$. Then $\mathbf{A}^{\lambda}$ is the spectrum of the ring $\mathrm{Sym}(\mathrm{Sym}^d)$, and thus identified with the dual space $(\mathrm{Sym}^d)^*$. This, in turn, is naturally identified with the degree d piece of the inverse limit ring $\mathbf{R}$. To be completely concrete, a point in $\mathbf{A}^{(d)}$ can be written in the form $\sum_{|\alpha|=d} c_{\alpha} x^{\alpha}$, where the sum is over exponent vectors α and $c_{\alpha} \in k$, and a closed subset is defined by polynomial equations in the coefficients (the c's).

Example 12.4. Let $\underline{\lambda} = [(d_1), \ldots, (d_r)]$. Then $\mathbf{A}^{\underline{\lambda}}$ parametrizes tuples $(f_1, \ldots, f_r) \in \mathbf{R}$ where f_i is homogeneous of degree d_i. Thus $\mathbf{A}^{\underline{\lambda}}$ is a kind of moduli space for (generators of) ideals in the Stillman regime. At each point in $\mathbf{A}^{\underline{\lambda}}$, one can consider the projective dimension of the corresponding ideal, and Stillman's conjecture roughly asserts that this function is bounded. Thus one might hope to prove Stillman's conjecture (and similar statements) by understanding aspects of **GL**-varieties. We pursue this idea in Lectures 15 and 16.

If you're puzzled by what a general $\mathbf{A}^{\lambda}$ looks like, you can simply focus on $\mathbf{A}^{(d)}$'s without losing too much. We now give some examples of more interesting **GL**-varieties.

Example 12.5. We have seen (Exercise 3.4) that for degree 2 elements in $\mathbf{R}$, the condition "strength $\leq s$" is described by polynomial equations on the coefficients.

Thus the strength $\leq s$ locus in $\mathbf{A}^{(2)}$ is a closed $\mathbf{GL}$-subvariety for any fixed s. (For elements of $\mathbf{A}^{(2)}$, rank and strength are closely related. If we had instead used the rank $\leq r$ locus, the defining ideal would be $\mathfrak{a}_r(\mathbf{V})$, where $\mathfrak{a}_r$ is as in Example 11.2.)

EXAMPLE 12.6. One can consider the strength $\leq s$ locus in $\mathbf{A}^{(d)}$. It is known that for $d \geq 4$ this locus is not Zariski closed [BBOV]. Its Zariski closure is a closed $\mathbf{GL}$-subvariety of $\mathbf{A}^{(d)}$. The theory of $\mathbf{GL}$-varieties developed in [BDES] gives tools for understanding examples like this one.

Exercises

Exercise 12.1. Describe all closed $\mathbf{GL}$-subvarieties of $\mathbf{A}^{(1)} \times \mathbf{A}^{(1)}$.

Exercise 12.2. Show that the rank $\leq r$ loci account for all the non-empty proper closed $\mathbf{GL}$-subvarieties of $\mathbf{A}^{(2)}$.

EXAMPLE 12.7. Let $\underline{\lambda} = [(d_1), \ldots, (d_r)]$. Show that any map of $\mathbf{GL}$-varieties $\varphi\colon \mathbf{A}^{\underline{\lambda}} \to \mathbf{A}^{(e)}$ has the form

$$\varphi(f_1, \ldots, f_r) = \Phi(f_1, \ldots, f_r)$$

where $\Phi \in k[T_1, \ldots, T_r]$ is polynomial. Moreover, show that Φ is homogeneous of degree e if T_i is given degree d_i. (A map of $\mathbf{GL}$-varieties is simply a $\mathbf{GL}$-equivariant map of schemes over k.)

Additional exercises

Exercise 12.3. A point in a $\mathbf{GL}$-variety is $\mathbf{GL}$-*generic* if it has dense $\mathbf{GL}$-orbit.

(a) Show that any element of $\mathbf{A}^{(2)}$ of infinite strength is $\mathbf{GL}$-generic. In fact, this is true in $\mathbf{A}^{(d)}$ as well, but the proof is much harder.
(b) Let $f = \sum_{i \geq 0} x_{3i+1}x_{3i+2}x_{3i+3}$, regarded as a point of $\mathbf{A}^{(3)}$. Show that f is $\mathbf{GL}$-generic. (Hint: show that any cubic in n variables can be obtained as a limit of points in the $\mathbf{GL}$-orbit of f.)
(c) On the other hand, show that there is no point in $\mathrm{Sym}^3(k^n)$ with dense $\mathbf{GL}_n$-orbit. Thus the above result is somewhat surprising.
(d) Show that if $\underline{\lambda}$ is any tuple of non-empty partitions the space $\mathbf{A}^{\underline{\lambda}}$ admits a $\mathbf{GL}$-generic point.

Exercise 12.4. Let $X = \mathrm{Spec}(R)$ be an irreducible $\mathbf{GL}$-variety. The *invariant function field* of X is the field $k(X)^{\mathbf{GL}}$ of $\mathbf{GL}$-invariant elements in the function field $k(X) = \mathrm{Frac}(R)$. Compute this when X is the closed subvariety of $\mathbf{A}^{(1)} \times \mathbf{A}^{(1)}$ consisting of pairs (x, y) that are linearly dependent.

Notes

The notion of $\mathbf{GL}$-variety was formally introduced in [BDES, §2], though the idea had been in the air for a little while (e.g., [Dr] is about $\mathbf{GL}$-varieties but doesn't use that term).

LECTURE 13

The embedding and shift theorems

A common theme in representation stability is that objects can be made simpler by applying a "shift" operation. This is true of **GL**-varieties, as we will see in this lecture. These results give us an inductive way to understand **GL**-varieties, and are among the most powerful tools we have.

We first define the shift operation. Let $G(n)$ be the subgroup of $\mathbf{GL}_n$ consisting of matrices of the form

$$\begin{pmatrix} 1 & 0 \\ 0 & * \end{pmatrix},$$

where the top left block is $n \times n$. Of course, $G(n)$ is itself isomorphic to **GL**. Given an object X on which **GL** acts, we can thus restrict to $G(n)$ and then identify $G(n)$ with **GL** to obtain a new action of **GL**. This is called the nth *shift* of X, and denoted $\mathrm{Sh}_n(X)$. For example, we have $\mathrm{Sh}_n(\mathbf{V}) = \mathbf{C}^n \oplus \mathbf{V}$, and one can use this to figure out shifts of Schur functors, e.g.,

$$\mathrm{Sh}_n(\mathrm{Sym}^2(\mathbf{V})) = \mathrm{Sym}^2(\mathbf{C}^n \oplus \mathbf{V}) = \mathrm{Sym}^2(\mathbf{C}^n) \oplus (\mathbf{C}^n \otimes \mathbf{V}) \oplus \mathrm{Sym}^2(\mathbf{V}).$$

The following is the *embedding theorem*:

THEOREM 13.1. *Let Y be a* **GL***-variety, let λ be a non-empty partition, and let X be a closed* **GL***-subvariety of $Y \times \mathbf{A}^\lambda$. Then one of the following two possibilities holds:*

(a) $X = Y_0 \times \mathbf{A}^\lambda$ for some closed **GL***-subvariety $Y_0 \subset Y$; or*
(b) there is a non-empty open subset of $\mathrm{Sh}_n(X)$, for some n, that embeds into $\mathrm{Sh}_n(Y) \times \mathbf{A}^{\underline{\mu}}$ for some $\underline{\mu}$, where every partition in $\underline{\mu}$ is smaller than λ.

Before giving the proof, we illustrate the main idea in a special case.

EXAMPLE 13.2. Let Y be a point, let $\lambda = (2)$, and let X be the rank ≤ 1 locus in $\mathbf{A}^{(2)}$. We think of elements of $\mathbf{A}^{(2)}$ as infinite symmetric matrices x, and write $x_{i,j}$ for the (i,j) entry. Let $U \subset X$ be the open set where $x_{1,1}$ is non-zero; this is $G(1)$-stable. Suppose $x \in U$. Then x has rank 1, and the first column is a basis for its column space. For $i > 1$, the ith column of x is a scalar multiple of the first column, and by looking at the first row we see that the scalar is $x_{i,1}/x_{1,1}$. Thus we can solve for all entries in x in terms of the first row. This shows that

$$\mathrm{Sh}_1(U) \cong \mathbf{C}^\times \times \mathbf{A}^{(1)}$$

as **GL**-varieties. Here the $\mathbf{C}^\times$ is the $x_{1,1}$ coordinate, and the $\mathbf{A}^{(1)}$ records the first row with $x_{1,1}$ omitted. (Note that the first row with $x_{1,1}$ omitted looks like $\mathbf{A}^{(1)}$ but with $G(1)$-acting; after shifting, we actually get $\mathbf{A}^{(1)}$ with **GL** acting.)

Proof of Theorem 13.1. The main idea is like the example: we will construct a function h such that on the $h \neq 0$ locus we can solve for many of the coordinates in terms of simpler coordinates. This will produce an embedding of the kind we want. We argue as follows:

- We prove the theorem just for $\lambda = (2)$ for simplicity; the general argument is exactly the same, just with more complicated notation.
- Let R be the coordinate ring of Y, so that $R[x_{i,j}]$ is the coordinate ring of $Y \times \mathbf{A}^{\lambda}$ (where $x_{i,j} = x_{j,i}$). Let $I \subset R[x_{i,j}]$ be the ideal for X, let J_0 be its contraction to R, and let J be the extension of J. We have $J \subset I$ with equality if and only if we're in case (a). Assume we're not in case (a), so I is strictly larger than J.
- We have seen that polynomial representation of $\mathbf{GL}_{\infty}$ are determined by their 1^n weight spaces. Thus the 1^n weight space of I is strictly larger than that for J, for some n; let f be such a weight vector in I that's not in J.
- A 1^n weight vector in $R[x_{i,j}]$ can be written as a sum of terms of the form $x_{i_1,j_1} \cdots x_{i_r,j_r} g$ where all indices are distinct and g is an 1^S-weight vector of R, where $S = [n] \setminus \{i_1, j_1, \ldots, i_r, j_r\}$. Applying a permutation to f, we can thus assume that $f = hx_{n-1,n} + g$, where h is a non-zero 1^{n-2}-weight vector in $R[x_{i,j}]$ and the variable $x_{n-1,n}$ does not appear in g.
- In $(R[x_{i,j}]/I)[1/h]$, we have $x_{n-1,n} = -g/h$. The variables appearing in the right side are of the form $x_{i,j}$ or $x_{n,i}$ or $x_{n-1,i}$ where $i, j \leq n-2$; call these "small." Thus applying permutations of $\{n-1, n, n+1, \ldots\}$, we see that every $x_{i,j}$ can be expressed in terms of small variables in this ring. In other words, we have a $G(n)$-equivariant surjection

  $$(R[x_{i,j}]_{1 \leq i,j \leq n-2}[1/h]) \otimes k[y_i, z_i]_{i \geq n-1} \to (R[x_{i,j}]/I)[1/h]$$

  where y_i maps to $x_{n-1,i}$ and z_i to $x_{n,i}$. Thus case (b) holds with $\underline{\mu} = [(1), (1)]$.

$\square$

The *shift theorem* is the following. For a function h on a variety X, we let $X[1/h]$ be the non-vanishing locus of h.

THEOREM 13.3. *Let X be a $\mathbf{GL}$-variety. Then there is $n \geq 0$ and a non-zero $\mathbf{GL}$-invariant function h on $\mathrm{Sh}_n(X)$ such that $\mathrm{Sh}_n(X)[1/h]$ is isomorphic, as a $\mathbf{GL}$-variety, to $B \times \mathbf{A}^{\underline{\rho}}$, where B is an ordinary (finite dimensional) variety and $\underline{\rho}$ is a tuple of partitions.*

Proof. Embed X into $\mathbf{A}^{\underline{\mu}}$ for some $\underline{\mu}$. We proceed by induction on $\underline{\mu}$. If $\underline{\mu}$ only consists of empty partitions, the result is clear (we don't need to shift or pass to an open set: we can just take $B = X$ and $\underline{\rho}$ to be empty). This is the base case of the induction.

Suppose now that $\underline{\mu}$ contains some non-empty partition. Let N be the maximal size of a partition in $\underline{\mu}$, let λ be a partition in $\underline{\mu}$ of size N, let $\underline{\nu}$ be the reamining part of $\underline{\mu}$, and let $Y = \mathbf{A}^{\underline{\nu}}$. We have $X \subset Y \times \mathbf{A}^{\lambda}$, so we are in the setting of the embedding theorem. In case (i), we have $X = Y_0 \times \mathbf{A}^{\lambda}$ for some $Y_0 \subset \mathbf{A}^{\underline{\nu}}$. Since $\underline{\nu}$ is smaller than $\underline{\mu}$, the shift theorem holds for Y_0 by induction; it is easy to see that it then holds for X. Now suppose we're in case (ii). Then after shifting and passing to an open set, X embeds into $\mathrm{Sh}_n(Y) \times \mathbf{A}^{\underline{\sigma}}$, where every partition in $\underline{\sigma}$ is smaller than λ. This space has the form $\mathbf{A}^{\underline{\tau}}$, where $\underline{\tau}$ is smaler than $\underline{\mu}$. (All partitions in $\underline{\tau}$ have size at most N, and the number of partitions in $\underline{\tau}$ of size N is one less than the number in $\underline{\mu}$ of size N.) Thus by induction, the shift theorem holds for subvarieties of $\mathbf{A}^{\underline{\tau}}$, and so the result follows. $\square$

Exercises

Exercise 13.1. Let X be the closed **GL**-subvariety of $\mathbf{A}^{[(1),(1)]}$ consisting of linearly dependent pairs. Explicitly verify the conclusion of the shift theorem in this case.

Exercise 13.2. Let X be the rank $\le r$ locus in $\mathbf{A}^{(2)}$. Explicitly verify the conclusion of the shift theorem in this case.

Exercise 13.3. Show that $\mathrm{Sh}_n(\mathbf{S}_\lambda)$ has the form $\mathbf{S}_\lambda \oplus \cdots$, where the remaining terms are Schur functors of smaller degree.

Additional exercises

Exercise 13.4. Let X be a **GL**-variety. Show that the invariant function field $k(X)^{\mathbf{GL}}$ is a finitely generated extension of k. (See Exercise 12.4 for the definition of $k(X)^{\mathbf{GL}}$.) [Hint: use the shift theorem (Theorem 13.3).]

Exercise 13.5. Explicitly compute $\mathrm{Sh}_n(\mathbf{S}_\lambda)$ in terms of Littlewood–Richardson coefficients (if you know what these are).

Exercise 13.6. Let X be an affine **GL**-variety.

(a) Show that there is a natural surjective map of **GL**-varieties $\mathrm{Sh}_n(X) \to X$. [Hint: this is induced by the canonical inclusion $\mathbf{V} \to \mathrm{Sh}_n(\mathbf{V})$.]
(b) Show that there is a dominant morphism $B \times \mathbf{A}^{\underline{\lambda}} \to X$ for some finite dimensional variety B and some tuple $\underline{\lambda}$. [This says that X is "unirational up to a finite dimensional error."]

Notes

The embedding theorem appeared implicity in [Dr]. It was isolated as a standalone result in [BDES, §4] when it was realized how useful it can be. The shift theorem was proved in [BDES, §5]. See [BDES] for more details on the proofs.

Lecture 14

Draisma's theorem

Recall that a topological space is *noetherian* if it satisfies DCC on closed subsets. If X is an algebraic variety then its underlying topological space (with the Zariski topology) is noetherian; this is a weak version of the Hilbert basis theorem.

If X is a **GL**-variety then its underlying topological space is essentially never noetherian, for if we ignore the **GL** action then we can get things like infinite affine space. So if we want some finiteness, we need to take into account the group action. This is what the following definition does:

Definition 14.1. Suppose a group G acts on a topological space X. We say that X is *(topologically) G-noetherian*[3] if it satisfies DCC on G-stable closed subsets.

Example 14.2. The **GL**-variety $\mathbf{A}^{(2)}$ is **GL**-noetherian. Indeed, for $s \geq 0$ let $Z_s \subset \mathbf{A}^{(2)}$ be the rank $\leq s$ locus. We have seen (Exercise 12.2) that the Z_s are closed and that they account for all non-empty proper closed **GL**-subvarieties. We thus see that the lattice of closed **GL**-subvarieties of $\mathbf{A}^{(2)}$ is a single chain:

$$\varnothing \subset Z_0 \subset Z_1 \subset \cdots \subset Z_s \subset \cdots \subset \mathbf{A}^{(2)}.$$

This clearly satisfies DCC.

In 2017, Draisma proved the following important theorem, which vastly generalizes the above example:

Theorem 14.3. *An affine* **GL***-variety is* **GL***-noetherian.*

Proof. Draisma's proof is a rather involved induction argument. We present the main idea in a special case to circumvent much of inductive baggage. Specifically, we show that $\mathbf{A}^{\lambda}$ is **GL**-noetherian for a non-empty partition λ, assuming that $\mathbf{A}^{\underline{\mu}}$ is **GL**-noetherian whenever every partition in $\underline{\mu}$ is smaller than λ. Here is the argument:

- Given a proper closed **GL**-subvariety X of $\mathbf{A}^{\lambda}$, let δ_X be the minimal degree of an element of its ideal. We'll show that every proper closed **GL**-subvariety X of $\mathbf{A}^{\lambda}$ is **GL**-noetherian by induction on δ_X, which clearly implies that $\mathbf{A}^{\lambda}$ is **GL**-noetherian. The base case $\delta_X = 0$ is trivial (for then $X = \varnothing$).
- Thus suppose $\delta_X > 0$ and let f be an element of the ideal of X of degree δ_X. By the embedding theorem, there is a non-zero invariant function h on $\mathrm{Sh}_n(X)$, for some n, such that $\mathrm{Sh}_n(X)[1/h]$ embeds into $\mathbf{A}^{\underline{\mu}}$ for some tuple $\underline{\mu}$ in which all parts are smaller than λ. By the *proof* of the embedding theorem, we can take h to have degree strictly smaller than that of f.

[3] In many cases, X will be a scheme, and then there might be stronger versions of G-noetherianity one is interested in; this is why the word "topologically" is sometimes used.

- By assumption, $\mathbf{A}^{\underline{\mu}}$ is $\mathbf{GL}$-noetherian. It follows that the closed subvariety $\mathrm{Sh}_n(X)[1/h]$ is $\mathbf{GL}$-noetherian by Exercise 14.1(a). This means that $X[1/h]$ is $G(n)$-noetherian. By Exercise 14.1(c), we see that $U = \bigcup_{g \in \mathbf{GL}} gX[1/h]$ is $\mathbf{GL}$-noetherian. (Note that U is a quasi-affine $\mathbf{GL}$-variety.)
- Let $Z \subset X$ by the common zero locus of the $\mathbf{GL}$-orbit of h. This is a closed $\mathbf{GL}$-subvariety of $\mathbf{A}^{\lambda}$ with $\delta_Z < \delta_X$. Thus, by the inductive hypothesis, it is $\mathbf{GL}$-noetherian.
- We have $X = Z \cup U$. Since Z and U are both $\mathbf{GL}$-noetherian, so is X by Exercise 14.1(b).

We make some comments on the general case:

- One shows that $\mathbf{A}^{\underline{\lambda}}$ is $\mathbf{GL}$-noetherian by induction on the size of $\underline{\lambda}$ (in a certain sense).
- Write $\underline{\lambda} = \underline{\mu} \cup \nu$ where ν is a partition in $\underline{\lambda}$ of maximal size. By the inductive hypothesis, $\mathbf{A}^{\underline{\mu}}$ is $\mathbf{GL}$-noetherian.
- Let $\pi \colon \mathbf{A}^{\underline{\lambda}} \to \mathbf{A}^{\underline{\mu}}$ be the projection map. We show that $\pi^{-1}(Z)$ is $\mathbf{GL}$-noetherian for $Z \subset \mathbf{A}^{\underline{\mu}}$ a closed $\mathbf{GL}$-subvariety, proceeding by noetherian induction on Z. In other words, if we fix Z we can assume that for any proper closed $Z' \subset Z$ we already know that $\pi^{-1}(Z')$ is $\mathbf{GL}$-noetherian.
- Fix Z. Given $X \subset \pi^{-1}(Z)$, we define δ_X to be the smallest degree of an element of the ideal of X inside of the coordinate ring of $\pi^{-1}(Z)$. One shows that X is $\mathbf{GL}$-noetherian by induction on δ_X, similar to the previous argument. □

REMARK 14.4. We expect that $\mathbf{GL}$-varieties should satisfy stronger noetherian conditions. For example, ACC should hold for all closed $\mathbf{GL}$-subschemes (even non-reduced ones). This is not known yet though.

Exercises

Exercise 14.1. We establish some basic properties of equivariant noetherianity. In what follows, X is a space on which G acts, and all subsets are endowed with the subspace topology.

(a) Suppose X is G-noetherian and A is a G-stable subset. Show that A is G-noetherian.
(b) Suppose $X = A \cup B$ where A and B are G-stable and G-noetherian. Show that X is G-noetherian.
(c) Suppose H is a subgroup of G and Y is an H-stable subset of X that is H-noetherian. Show that $\bigcup_{g \in G} gY$ is G-noetherian.

Exercise 14.2. Let X be a closed $\mathbf{GL}$-subvariety of $\mathbf{A}^{\underline{\lambda}}$. Show that there are finitely many functions $f_1, \ldots, f_r$ on $\mathbf{A}^{\underline{\lambda}}$ such that X is the common zero locus of the $\mathbf{GL}$-orbits of the f_i's.

Additional exercises

Exercise 14.3. Let R be the coordinate ring of $\mathbf{A}^{(1)} \times \mathbf{A}^{(1)}$. Show that $\mathbf{GL}$-ideals in R satisfy ACC.

Exercise 14.4. Let $\mathfrak{S}$ be the infinite symmetric group (whichever version you prefer), let $\mathfrak{S}$ act on $R = \mathbf{C}[x_1, x_2, \ldots]$ by permuting variables, and let $X = \operatorname{Spec}(R)$. Show that X is $\mathfrak{S}$-noetherian. [Warning: I do not know an easy proof!]

Notes

Draisma proved his theorem in [Dr]. That such a statement might be true seems to be first suggested in [Sn, §6]. The case of $\mathbf{A}^{(3)}$ was treated earlier by Derksen, Eggermont, and Snowden [DES], following prior work of Eggermont [Eg] in the degree two case.

Lecture 15

Stillman's conjecture via GL-varieties

In this lecture, we sketch a geometric proof of Stillman's conjecture based on Draisma's theorem. For a k-algebra A, let $\mathbf{R}_A$ be the inverse limit of the rings $A[x_1, \ldots, x_n]$ in the category of graded rings, where A is concentrated in degree 0. If $A \to B$ is a homomorphism of k-algebras, there is an induced homomorphism $\mathbf{R}_A \to \mathbf{R}_B$. In particular, if M is a graded $\mathbf{R}_A$-module and x is a point of $\mathrm{Spec}(A)$ then $M_x = M \otimes_A \kappa(x)$ is naturally a graded $\mathbf{R}_{\kappa(x)}$-module. Thus M gives rise to a family of $\mathbf{R}$-modules over $\mathrm{Spec}(A)$. We let $p_M(x)$ be the projective dimension of M_x as an $\mathbf{R}_{\kappa(x)}$-module.

Proposition 15.1. *Let M be a finitely presented $\mathbf{R}_A$-module, and suppose A is an integral domain. Then there is a dense open set U of $\mathrm{Spec}(A)$ such that $p_M(x)$ is constant for $x \in U$.*

Proof. Let $K = \mathrm{Frac}(A)$. The ring $K \otimes_A \mathbf{R}_A$ is a polynomial ring (Exercise 15.3). It follows that $K \otimes_A M$ has finite projective dimension over $K \otimes_A \mathbf{R}_A$ (Exercise 15.1). Now one "spreads out" the resolution over $\mathrm{Spec}(A)$; see Exercise 15.4 for how this works in a simpler case. □

Fix positive integers $d_1, \ldots, d_r$. Let A be the **GL**-algebra

$$A = \mathrm{Sym}(\mathrm{Sym}^{d_1}(\mathbf{V}) \oplus \cdots \oplus \mathrm{Sym}^{d_r}(\mathbf{V})).$$

Note that $\mathrm{Spec}(A)$ is exactly the **GL**-variety $\mathbf{A}^{\underline{\lambda}}$ considered in Example 12.4. Explicitly, A is the polynomial ring in variables $c_{i,\alpha}$, where $1 \le i \le r$, and α varies over exponent vectors of degree d_i (so that x^α varies over degree d_i monomials). Define F_i to be the element of $\mathbf{R}_A$ given by $F_i = \sum_{|\alpha|=d_i} c_{i,\alpha} x^\alpha$. Thus $(F_1, \ldots, F_r)$ is a universal tuple of degree $(d_1, \ldots, d_r)$, in that any such tuple in $\mathbf{R}$ can be obtained from this one along a base change $A \to k$. Let $M = \mathbf{R}_A/(F_1, \ldots, F_r)$.

Theorem 15.2. *The space $\mathrm{Spec}(A)$ admits a finite decomposition $\bigsqcup_{i=1}^n K_i$ where each K_i is a locally closed **GL**-subvariety of $\mathrm{Spec}(A)$ such that p_M is constant on each K_i.*

Proof. Let $Z = \mathrm{Spec}(A')$ be a closed **GL**-subvariety of $\mathrm{Spec}(A)$. Applying Proposition 15.1 to $M \otimes_A A'$, we see that there is a dense open subset V_0 of Z such that p_M is constant on V_0. For $g \in \mathbf{GL}$, the modules M_x and M_{gx} have the same projective dimensions; in other words, the function p_M is **GL**-invariant. Thus, putting $U = \bigcup_{g \in \mathbf{GL}} gV_0$, we see that p_M is constant on V. The set V is an open dense **GL**-stable subset of Z.

The result now follows Draisma's theorem (Theorem 14.3). Indeed, let $Z_0 = \mathrm{Spec}(A)$. By the previous paragraph, there is a dense open $K_0 \subset Z_0$ such that p_M

is constant on K_0. Now put $Z_1 = Z_0 \setminus K_0$, which is a closed $\mathbf{GL}$-subvariety of Z_0. Applying the previous paragraph again, there is a dense open $K_1 \subset Z_1$ such that p_M is constant on K_1. Now put $Z_2 = Z_1 \setminus K_1$, and continue. The descending chain $Z_\bullet$ of closed $\mathbf{GL}$-subvarieties of $\mathrm{Spec}(A)$ stabilizes by Draisma's theorem, and it must stabilize at the empty set. □

COROLLARY 15.3. *There is an integer $N = N(d_1, \ldots, d_r)$ with the following property: if $f_1, \ldots, f_r \in \mathbf{R}$ have degrees $d_1, \ldots, d_r$ then the projective dimension of $\mathbf{R}/(f_1, \ldots, f_r)$ is at most N.*

Proof. This follows from the theorem since $\mathbf{R}/(f_1, \ldots, f_r)$ has the form M_x for some $x \in \mathrm{Spec}(A)$. □

Stillman's conjecture is in fact a special case of the above corollary:

COROLLARY 15.4. *If $f_1, \ldots, f_r$ are homogeneous elements of $R_n = k[x_1, \ldots, x_n]$ of degrees $d_1, \ldots, d_r$ then the projective dimension of $R_n/(f_1, \ldots, f_r)$ is at most the number N from Corollary 15.3.*

Proof. See Exercise 15.4. □

Exercises

Exercise 15.1. Let $\mathfrak{a}$ be a finitely generated ideal of polynomial ring (such as $\mathbf{R}$ or $K \otimes_A \mathbf{R}$). Show that $\mathrm{pdim}_{\mathbf{R}}(\mathfrak{a})$ is finite. [This is elementary: you don't need to use anything like Stillman's conjecture.]

Let A be an integral k-algebra with fraction field K.

Exercise 15.2. How are the rings $\mathbf{R}_K$ and $K \otimes_A \mathbf{R}_A$ related? [Is there a homomorphism? Is it injective/surjective/isomorphism?]

Exercise 15.3. Show that $K \otimes_A \mathbf{R}_A$ is a polynomial K-algebra. [Hint: use Theorem 4.3].

Additional exercises

Exercise 15.4. Regard the polynomial algebra $A[x_1, \ldots, x_n]$ as a graded ring where A has degree 0 and each x_i has degree 1. Suppose that M is a finitely presented graded A-module. Show that there is an open dense subset U of $\mathrm{Spec}(A)$ such that the Betti table of M_x is constant for $x \in U$.

[Hint: consider the resolution of $K \otimes_A M$ over $K[x_1, \ldots, x_n]$. The differentials are matrices with entires in K. They therefore belong to $A[1/f]$ for some non-zero $f \in A$. In this way, one can spread out the complex to $A[1/f]$. Further argument is needed to show that one can choose f so that the complex is exact, and that the projective dimension doesn't go down.]

Exercise 15.5. Deduce Corollary 15.4 from Corollary 15.3.

Notes

The proof in this lecture is based on the material in [ESS2, §5]. All of the technical details are treated there. For a more expository account (omitting many details), see [ESS3, §9].

LECTURE 16

Beyond Stillman's conjecture

Stillman's conjecture asserts that the invariant "projective dimension" is bounded for ideals in the "Stillman regime," where the number and degrees of generators is fixed. In this lecture, we will see that more general invariants of ideals are also bounded in the Stillman regime.

An *ideal invariant* is a rule ν assigning a quantity $\nu(I) \in \mathbf{N} \cup \{\infty\}$ to each homogeneous ideal I in a standard-graded polynomial ring $R = k[x_1, \dots, x_n]$ such that $\nu(I)$ only depends on the pair (R, I) up to isomorphism. In this context, an isomorphism $(R, I) \to (R', I')$ is an isomorphism $f \colon R \to R'$ of graded rings such that $f(I) = I'$; essentially, this just allows for linear changes in the variables. There are many examples of ideal invariants: projective dimension, regularity, the (i, j) Betti number, etc.

We say that an ideal invariant ν is *Stillman bounded* if for each tuple $\mathbf{d} = (d_1, \dots, d_r)$ of positive integers there is a quantity $C = C(\mathbf{d})$ such that whenever I is generated by r elements of degrees $d_1, \dots, d_r$ we have $\nu(I) \le C$ or $\nu(I) = \infty$.

It is hard to say anything meaningful about general ideal invariants. We will require two conditions on our ideal invariant ν to get some control. The first condition is fairly straightforward: we say that ν is *cone-stable* if $\nu(I) = \nu(I[x])$ for all (R, I); that is, adjoining a new variable does not change the invariant. Projective dimension is easily seen to be cone-stable.

The second condition is that $\nu(I)$ should be a continuous function of I, in a certain sense. This is a bit technical to formulate. We assume k is algebraically closed for simplicity. Let B be a finitely generated k-algebra and let $I \subset B[x_1, \dots, x_n]$ be a homogeneous ideal (with B in degree 0). Given a (closed) point s of $X = \operatorname{Spec}(B)$, i.e., a k-algebra homomorphism $s \colon B \to k$, let I_s be the extension of I along the homomorphism $B[x_1, \dots, x_n] \to k[x_1, \dots, x_n]$. We thus have a function

$$\text{(closed points of) } X \to \mathbf{N} \cup \{\infty\}, \qquad s \mapsto \nu(I_s).$$

We say that ν is *strongly upper semi-continuous* if this function is always upper semi-continuous. (Recall that "upper semi-continuous" means that the function can "jump up" as a limit point is approached.) This condition is quite strong, as most invariants only behave well in flat families. We say therefore say that ν is *weakly upper semi-continuous* if the above function is upper semi-continuous whenever $B[x_i]/I$ is flat as a B-module. Projective dimension is weakly upper semi-continuous: in fact, it is constant in flat families.

The following is the main theorem:

THEOREM 16.1. *Any ideal invariant that is cone-stable and weakly upper semi-continuous is Stillman bounded.*

Proof. We simply sketch a proof. Let ν be a cone-stable ideal invariant and fix a tuple $\mathbf{d} = (d_1, \ldots, d_r)$ of positive integers.

Let $X_{\mathbf{d},n} = \mathrm{Sym}^{d_1}(k^n) \times \cdots \times \mathrm{Sym}^{d_r}(k^n)$, regarded as a finite-dimensional affine variety. We have a natural inclusions $X_{\mathbf{d},n} \subset X_{\mathbf{d},n+1}$, and we let $X_{\mathbf{d}}$ be the direct limit. This is an ind-scheme (and not a scheme). The infinite general linear group $\mathbf{GL}$ naturally acts on $X_{\mathbf{d}}$. Draisma's theorem still holds in this context, though it requires non-trivial justification.

Each (closed) point s of $X_{\mathbf{d}}$ defines a finitely homogeneous generated ideal I_s of $k[x_1, x_2, \ldots]$. Since ν is cone-stable, it is well-defined on such ideals (just go to a finite variable ring where all the generators live). We therefore have a well-defined function

$$X_{\mathbf{d}} \to \mathbf{N} \cup \{\infty\}, \qquad s \mapsto \nu(I_s)$$

Let $Z_n \subset X_{\mathbf{d}}$ be the set of points s such that $\nu(I_s) \geq n$. The Z_n's form a descending chain of subsets of $X_{\mathbf{d}}$.

Suppose that ν is strongly upper semi-continuous. This exactly means that the Z_n's are closed. By Draisma's theorem, the Z_n's stabilize. If $Z_n = Z_{n_0}$ for $n > n_0$ then we must have $\nu(I_s) \leq n_0$ or $\nu(I_s) = \infty$ for all s, and so ν is Stillman bounded.

Now suppose that ν is only weakly upper semi-continuous. Using the usual form of Stillman's conjecture, one shows that there is a finite locally closed stratification of $X_{\mathbf{d}}$ such that the family $\{k[x_i]/I_s\}$ is flat on each stratum. One can then argue as in the previous paragraph. $\square$

While Theorem 16.1 appears to be quite general, it is actually rather difficult to find examples of ideal invariants to which it applies. Here's one:

Example 16.2. Fix an integer $c \geq 1$. Given a homogeneous ideal $I \subset k[x_1, \ldots, x_n]$, consider the space X_I of codimension c linear of $\mathbf{A}^n$ contained in $V(I)$; this is naturally a scheme, in fact, a closed subscheme of the appropriate Grassmannian. We define $\nu(I)$ as follows: if X_I is a finite collection of reduced points then $\nu(I) = \#X_I$; otherwise $\nu(I) = \infty$. One can use the theorem to show that ν is Stillman bounded (though it is not quite cone-stable). This example is inspired by the classical fact that a smooth cubic surface has 27 lines.

Remark 16.3. Let $Y_{\mathbf{d}}$ be the set of isomorphism classes of homogeneous ideals of $k[x_1, x_2, \ldots]$ generated by r elements of degrees $d_1, \ldots, d_r$. There is a surjective $\mathbf{GL}$-invariant map $X_{\mathbf{d}} \to Y_{\mathbf{d}}$. Giving $Y_{\mathbf{d}}$ the quotient topology, it follows from Draisma's theorem that $Y_{\mathbf{d}}$ is a noetherian topological space.

Exercises

Exercise 16.1. Give an example of an ideal invariant that is not Stillman bounded.

Exercise 16.2. Verify directly that projective dimension is cone-stable and weakly upper semi-continuous.

Notes

This lecture is based on [ESS1]. For ideal invariants that are strongly upper semi-continuous, one can improve Theorem 16.1; see [ESS1, §5.2].

LECTURE 17

Ultrahomogeneous forms

In what we have done so far, all infinite strength elements of $\mathbf{R}$ have been more or less equal: for example, all obey Principle 1.7 and are all $\mathbf{GL}$-generic (see Exercise 12.3). It is therefore reasonable to ask: are there any interesting features that some infinite strength forms have that others do not? In this lecture, we see that the answer is a resounding yes!

To motivate our discussion, let us first return to the finite variable setting. All non-degenerate quadratic forms in n-variables over the complex numbers are equivalent, i.e., belong to the same $\mathbf{GL}_n$-orbit. There is no statement like this for cubic forms: cubic forms in n variables are a zoo, with ever more creatures as n grows. One might therefore think the infinite dimensional case would be hopeless.

There is a roughly similar situation with graphs, in that graphs are also a zoo that becomes increasing complex as the number of vertices increasses. However, it turns out that, rather amazingly, there is a "best" infinite graph:

THEOREM 17.1. *There exists a countable graph Γ, called the* Rado graph*, having the following two properties:*

(a) Universality: every finite graph occurs as an induced subgraph of Γ.
(b) Ultrahomogeneity: any isomorphism $\Delta \to \Delta'$ of finite induced subgraphs of Γ extends to an automorphism of Γ.

Moreover, Γ is unique up to isomorphism.

The ultrahomogeneity condition implies that the automorphism group G of the Rado graph is quite large. Indeed, let V be the vertex set of the Rado graph, and write $V^{(n)}$ for the set of n-element subsets of V. Then the orbits of G on $V^{(n)}$ are in bijective correspondence with isomorphism classes of graphs with n vertices (Exercise 17.2). In particular, G has finitely many orbits on $V^{(n)}$. A permutation group with this property is called *oligomorphic*.

It turns out that there is an analog analog of Theorem 17.1 for forms. To state this, it will be convenient to introduce some terminology. A *symmetric d-form* on a vector space V is a linear map $\mathrm{Sym}^d(V) \to \mathbf{C}$. A *symmetric d-space* is a vector space equipped with a symmetric d-form. For example, a symmetric 2-space is a quadratic space, i.e., a vector space with a quadratic form. There are obvious notions of isomorphism and embedding for symmetric d-spaces. We can now state our theorem:

THEOREM 17.2. *Fix $d \geq 1$. There exists a symmetric d-space V of countable dimension having the following two properties:*

(a) Universality: every finite dimensional symmetric d-space embeds into V.

(b) Ultrahomogeneity: if W and W' are finite dimensional subspaces of V equipped with the induced forms, then any isomorphism $W \to W'$ extends to an automorphism of V.

Moreover, V is unique up to isomorphism.

In other words, of the infinite strength degree d elements of $\mathbf{R}$, there is a unique isomorphism class of ultrahomogeneous forms; simply put, there is a "best" degree d form! As with the Rado graph, the automorphism group of this form is huge.

The above theorems can be proved using a construction called the *Fraïssé limit*. We provide a very abridged account. A *finite relational structure* is a finite set equipped with a number of relations (of possibly varying arities); for example, one can think of a finite graph as a finite relational structure with a single binary relation. Let $\mathcal{C}$ be a class of finite relational structures. We consider the following two conditions:

(JEP) We say that $\mathcal{C}$ has the *joint embedding property* if any two members of $\mathcal{C}$ embed into some other member.

(AP) We say that $\mathcal{C}$ has the *amalgamation property* if, given embeddings $W \to X$ and $W \to Y$ in $\mathcal{C}$, one can find a commutative square

$$\begin{tikzcd} X \arrow[r] & Z \\ W \arrow[u] \arrow[r] & Y \arrow[u] \end{tikzcd}$$

of embeddings in $\mathcal{C}$.

Notice that if $\mathcal{C}$ has an initial member then (JEP) is a special case of (AP). We can now state Fraïssé's theorem:

THEOREM 17.3. *Suppose that $\mathcal{C}$ satisfies (AP) and (JEP) and has countably many members (up to isomorphism). Then there is a countable structure, constructed as a union of a chain in $\mathcal{C}$, that is universal (all elements of $\mathcal{C}$ embed into it) and ultrahomogeneous. It is unique, up to isomorphism.*

It is easy to deduce Theorem 17.1 from Theorem 17.3. It is also not too hard to deduce Theorem 17.2 from Theorem 17.3 when the coefficient field is countable. In general, one needs a more powerful variant of Theorem 17.3.

Exercises

Exercise 17.1. Let $\mathcal{C}$ be the class of finite graphs. Show that $\mathcal{C}$ satisfies (AP). [Note that the empty graph is initial, and so (JEP) holds as well.]

Exercise 17.2. Let G be the automorphism group of the Rado graph, let V be the vertex set of the Rado graph, and let $V^{(n)}$ be the set of n-element subsets of V. Show that the G-orbits on $V^{(n)}$ are in natural bijection with isomorphism classes of graphs on n vertices.

Exercise 17.3. Show that $(\mathbf{Q}, <)$ is ultrahomogeneous, as a totally ordered set. It is thus the Fraïssé limit of the class of finite totally ordered sets.

Exercise 17.4. Work over a finite field $\mathbf{F}$ of characteristic $\neq 2, 3$. Explain how a cubic space can be encoded as a relational structure. Let $\mathcal{C}$ be the class of finite dimensional cubic spaces over $\mathbf{F}$. Show that $\mathcal{C}$ satisfies (AP). [Note that the zero space is initial, and so (JEP) holds as well.]

Notes

Theorem 17.2, and some generalizations and related material, will appear in [HS]. For background on Fraïssé limits, I highly recommend Cameron's book [Cam].

LECTURE 18

Universality of strength

In the previous lecture, we saw that there is an essentially unique degree d form that is both ultrahomogeneous and universal. One might therefore wonder about forms having just one of these properties. In this lecture, we consider the universal case. The main result is the following theorem:

THEOREM 18.1. *Let V be a symmetric d-space of countable infinite dimension for which the defining form has infinite strength. Then V is universal, i.e., any finite dimensional symmetric d-space embeds into V.*

This theorem can be reformulated in the language of forms as follows:

THEOREM 18.2. *Let f be a homogeneous element of $\mathbf{R}$ of degree d and infinite strength. Given any n and any homogeneous g of R_n of degree d, there exists a continuous homomorphism $\varphi\colon \mathbf{R} \to R_n$ such that $\varphi(f) = g$.*

In the above theorem, "continuous" is taken with respect to the inverse limit topology on $\mathbf{R}$. Concretely, this simply means that $\varphi(x_i) = 0$ for all but finitely many i, and that φ commutes with infinite sums.

This theorem was first proved by Kazhdan–Ziegler [KaZ]. It was then generalized to arbitary polynomial functors by Bik, Danelon, Draisma, and Eggermont [BDDE]. The theorem is deduced from a more general result, which we now explain.

Let $X = \mathrm{Spec}(R)$ be a $\mathbf{GL}$-variety. By definition, R is a polyomial representation of $\mathbf{GL}$, which means that it decomposes into a sum of Schur functors $\mathbf{S}_{\underline{\lambda}}(\mathbf{V})$. However, since Schur functors are functors, *any* linear endomorphism of $\mathbf{V}$ induces an endomorphism of $\mathbf{S}_{\underline{\lambda}}(\mathbf{V})$. It follows that the monoid $\mathrm{End}(\mathbf{V})$ acts on R, and therefore on X. The following is the more general result:

THEOREM 18.3. *Let S be a* subset *of (the $\mathbf{C}$-points of) $\mathbf{A}^{(d)}$ that is closed under the action of* $\mathrm{End}(\mathbf{V})$. *Then exactly one of the following two possibilites holds:*

(a) S is contained in the strength $\le s$ locus, for some s.
(b) S contains every degree d polynomial.

In the above theorem, we identify $\mathbf{A}^{(d)}$ with the degree d part of the inverse limit ring $\mathbf{R}$. Thus a point in $\mathbf{A}^{(d)}$ is a formal linear combination of degree d monomials in the variables $\{x_i\}_{i \ge 1}$. We say that a point of $\mathbf{A}^{(d)}$ is a polynomial if it is a finite sum of monomials.

We now explain how to obtain Theorem 18.2 from Theorem 18.3. Let f be a degree d element of $\mathbf{R}$ of infinite strength. Let $X \subset \mathbf{A}^{(d)}$ be the $\mathrm{End}(\mathbf{V})$-orbit of f, i.e., $S = \{\sigma \cdot f \mid \sigma \in \mathrm{End}(\mathbf{V})\}$. Clearly, S is not contained in any strength $\le s$ locus, and so S contains every degree d polynomial by Theorem 18.3. Let $g \in R_n$

be of degree d. We can then find $\sigma \in \mathrm{End}(\mathbf{V})$ such that $\sigma \cdot f = g$. Using σ, one produces a continuous homomorphism $\varphi \colon \mathbf{R} \to R_n$ such that $\varphi(f) = g$. This proves Theorem 18.2.

Exercises

There are no exercises for the final lecture!

Notes

Theorem 18.2 was proved for $d = 3$ by Derksen, Eggermont, and Snowden [DES], and in general by Kazhdan and Ziegler [KaZ]. The statement was generalized to arbitrary polynomial functors by Bik, Danelon, Draisma, and Eggermont [BDDE]..

LECTURE 19

The decomposition theorem

I prepared this lecture for the summer school, but decided to replace it with the lecture on ultrahomogeneous forms. I have included the notes since they were already prepared.

By combining the shift theorem (Theorem 13.3) and Draisma's theorem (Theorem 14.3), we can obtain a powerful structural result for **GL**-varieties and morphisms of **GL**-varieties. This is called the *decomposition theorem*. In this lecture, we'll formulate the statements of these results and give some corollaries. The proofs are not too hard given the tools we have, and some parts are in the exercises.

DEFINITION 19.1. Let X be a quasi-affine **GL**-variety.

- X is *elementary* if it has the form $B \times \mathbf{A}^{\underline{\lambda}}$ where B is an irreducible affine **GL**-variety and $\underline{\lambda}$ is a tuple of partitions; note that elementary varieties are affine.
- A $G(n)$-variety is *elementary* if it is so after identifying $G(n)$ with **GL**. (Recall that $G(n)$ is the subgroup of **GL** introduced in Lecture 13.)
- X is *locally elementary* if it is a union of open $G(n)$-elementary subvarieties. (One can allow n to vary, but this is actually unnecessary.)
- A *locally elementary decomposition* (LED) of X is a decomposition $X = \bigsqcup_{i=1}^{n} X_i$ where each X_i is a locally closed **GL**-subvariety that is locally elementary.

The following is the decomposition theorem for **GL**-varieties:

THEOREM 19.2. *Every* **GL***-variety admits an LED.*

We now formulate the relevant definitions for morphisms:

DEFINITION 19.3. Let $\varphi \colon X \to Y$ be a morphism of quasi-affine **GL**-varieties.

- We say that φ is *elementary* if X and Y are elementary, and we can choose identifications $X = B \times \mathbf{A}^{\underline{\lambda}}$ and $Y = X \times \mathbf{A}^{\underline{\mu}}$, with $\underline{\mu} \subset \underline{\lambda}$, such that φ has the form $\psi \times \pi$, where $\psi \colon B \to C$ is a surjection of varieties and $\pi \colon \mathbf{A}^{\underline{\lambda}} \to \mathbf{A}^{\underline{\mu}}$ is the projection map.
- We say that a map of $G(n)$-varieties is *elementary* if it is so after identifying $G(n)$ with **GL**.
- We say that φ is *locally elementary at $x \in X$* if there are $G(n)$-stable open neighborhoods U of x and V of $\varphi(x)$ such that $\varphi \colon U \to V$ is elementary. We say that φ is *locally elementary* if it is surjective and locally elementary at all points of X.

- A *locally elementary decomposition* (LED) for φ consists of LEDs $X = \bigsqcup_{i=1}^n X_i$ and $Y = \bigsqcup_{j=1}^m Y_j$ such that for each i there is some j such that $\varphi(X_i) \subset Y_j$, and $\varphi \colon X_i \to Y_j$ is locally elementary.

The following is the decomposition theorem for morphisms of **GL**-varieties:

THEOREM 19.4. *Every morphism of* **GL***-varieties admits an LED.*

REMARK 19.5. Theorem 19.4 applied to the identity morphism recovers Theorem 19.2.

As a corollary of the decomposition theorem, we can prove an analog of Chevalley's theorem for **GL**-varieties, as follows. To state this theorem, we introduce another definition:

DEFINITION 19.6. Let X be a **GL**-variety. A subset of X is **GL**-*constructible* if it is a finite union of locally closed **GL**-subvarieties.

We can now state our version of Chevalley's theorem:

THEOREM 19.7. *Let $\varphi \colon X \to Y$ be a map of quasi-affine* **GL***-varieties and let K be a* **GL***-constructible subset of X. Then $\varphi(K)$ is a* **GL***-constructible subset of Y.*

Proof. The key case for the proof is $K = X$, and this follows directly from the decomposition theorem: since locally elementary maps are by definition surjective, the image of φ is the union of some collection of the Y_j's, and therefore **GL**-constructible. $\square$

Here is an interesting application of the theorem:

COROLLARY 19.8. *The strength s locus in $\mathbf{A}^{(d)}$ is* **GL***-constructible.*

Proof. See Exercise 19.3. $\square$

Exercises

Exercise 19.1. Let λ be a partition. Prove that $\mathbf{A}^\lambda \setminus \{0\}$ is locally elementary.

Exercise 19.2. Prove the decomposition theorem for **GL**-varieties (Theorem 19.2). [Hint: using the shift theorem, show that a non-empty **GL**-variety has a non-empty open **GL**-subvariety that is locally elementary; then proceed by noetherian induction.]

Exercise 19.3. Prove Corollary 19.8: the strength s locus in $\mathbf{A}^{(d)}$ is **GL**-constructible.

Additional exercises

Exercise 19.4. Let $\underline{\lambda}$ and $\underline{\mu}$ be tuples of non-empty partitions, and let $\varphi\colon \mathbf{A}^{\underline{\lambda}} \to \mathbf{A}^{\underline{\mu}}$ be a dominant morphism of $\mathbf{GL}$-varieties. Show that $\underline{\mu} \subset \underline{\lambda}$. (This is a key component in the proof of the decomposition theorem for morphisms.)

Exercise 19.5. Prove Theorem 19.7 in general.

Exercise 19.6. Let $\varphi\colon Y \to X$ be a map of $\mathbf{GL}$-varieties, and let x be a point in the image of φ. Show that there exists a pre-image y of x such that $\kappa(y)$ is a finite extension of $\kappa(x)$.

Exercise 19.7. Fix positive integers s and d. Show that there is a polynomial time algorithm that takes as input a positive integer n and a degree d polynomial $f \in \mathbf{Q}[x_1, \ldots, x_n]$, and determines if f has strength s. In fact, show that one can make the algorithm return a suitable decomposition $f = \sum_{i=1}^{s} g_i h_i$ when f does have strength s.

Notes

The decomposition theorem originally appears in [BDES, §7].

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