Exercises

1. Recall that we defined $h^+_i := x_i \partial_i$, $h^-_i := \partial_i x_i = h^+_i + 1$, and $[h_i]^k := \begin{cases} 1 & \text{if } k = 0 \\ x_i^k \partial_i^k = (h^-_i - 1) \cdots (h^-_i - k) & \text{if } k > 0 \\ \partial_i^{-k} x_i^{-k} = (h^+_i + 1) \cdots (h^+_i - k) & \text{if } k < 0. \end{cases}$

We also defined $m_z := x^z \partial^z$ for all $z \in \mathbb{Z}^n$, and we asserted that we have the following formula:

$$m_z \cdot m_w = \left( \prod_{z_i w_j < 0 \atop |z_i| \leq |w_j|} [h_i]^{z_i} \right) m^{z+w} \left( \prod_{z_i w_j < 0 \atop |z_i| > |w_j|} [h_i]^{-w_i} \right).$$

Let $n = 8$, $z = (1, 5, -2, -3, -4, 6, 0, 3)$, and $w = (-2, -3, 4, 1, -7, 2, 5, -3)$. Write down $m_z$ and $m_w$, and verify the formula in this case. (This is a situation where working out a single example is more illuminating than writing down a general proof.)

2. Let $T = \mathbb{C}^\times$, and consider the homomorphism $(\mathbb{C}^\times)^3 \to T$ taking $(t_1, t_2, t_3)$ to $t_1 t_2 t_3$. Let $K \subset (\mathbb{C}^\times)^3$ be the kernel of this homomorphism. Let $U := D^K$ be the corresponding hypertoric enveloping algebra.

i) For each $k \in \mathbb{Z} \cong \mathfrak{t}_2^* \subset \mathbb{Z}^3$, write down the element $m^k \in D^K = U$.

ii) Compute $m_3 \cdot m^{-4}$.

iii) What is the center of $U$?