QUANTIZATION EXERCISES

(1) The special linear group $SL_2$ acts on $\mathbb{P}^1$ by
\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot [x_0, x_1] = [ax_0 + bx_1, cx_0 + dx_1].
\]

(a) By differentiating this action action construct Lie algebra anti-homomorphism
\[
\alpha : \mathfrak{sl}_2 \rightarrow \Gamma(\mathbb{P}^1, T_{\mathbb{P}^1})
\]
This is an anti-homormophism because the standard Lie algebra structure on $\Gamma(X, T_X)$ is the opposite of the one you get from the identification
\[
\text{Lie}(\text{Aut}(X)) \cong \Gamma(X, T_X).
\]
(b) Show that $-\alpha$ induces a surjection $\eta : U(\mathfrak{sl}_2) \cong \Gamma(\mathbb{P}^1, D_{\mathbb{P}^1})$.
(c) Show that the Casimir $\frac{1}{2}h^2 + ef + fe$ goes to a scalar under $\eta$.
(d) The classical limit of $\eta$ gives a map $T^*\mathbb{P}^1 \rightarrow (\mathfrak{sl}_2)^*$. Show that the image of this map is the nilpotent cone. You will need to use the Killing form to identify $\mathfrak{sl}_2$ with it’s dual.

(2) A similar technique can be used to construct a map from $U(\mathfrak{sl}_2)$ to the hypertoric enveloping algebra $U$ for $G = C^*_\Delta \rightarrow (C^*)^2$. Recall that the central quotients of $U$ are parameterized by points $\lambda \in g^*$.
(a) Find a formula that gives the value of the Casimir in each $U_\lambda$.
(b) Which value of $\lambda$ gives the unique self opposite quantization? Draw the corresponding quantum arrangement.