WARTHOG 2017

Toric and Hypertoric Varieties

Exercises

1. Let $I = 1, 2, 3$, $V = \{(a, b, c) \in \mathbb{R}^3 \mid a - b + c = 0\}$, and $\eta = [1, 0, 0] \in \mathbb{R}^3/V$. Show that the toric variety $X(\mathcal{V})$ is isomorphic to the blow-up of $\mathbb{C}^2$ at the origin. You can do this either by expressing it as Proj of a graded ring, or by computing the stable locus in $\mathbb{C}^3$ and taking the quotient by the subtorus $K \subset (\mathbb{C}^\times)^3$.

2. Using the same $\mathcal{V}$, describe the hypertoric variety $Y(\mathcal{V})$. Show that it is a rank 2 vector bundle over $\mathbb{C}P^2$ (it is in fact the cotangent bundle). What are the extended core components? Describe the relative core when $\xi = [1, 2, 3] \in (\mathbb{R}^3)^*/V^\perp \cong V^*$.

3. Let $\mathcal{V}^!$ be the Gale dual of the polarized arrangement in Problems 1 and 2. Convince yourself that $Y(\mathcal{V}^!)$ is 2-dimensional and contains two copies of $\mathbb{C}P^1$ in its extended core.

(Hint: You don't really need to describe $Y(\mathcal{V}^!)$ in any detail, though it's fun and worthwhile to do so if you have the time. It's enough to draw the hyperplane arrangement in $V^\perp_{-\xi}$ and use what we know about the relationship between the chambers of this arrangement and the components of the extended core.)

In particular, this implies that $Y(\mathcal{V}^!)$ is not a cotangent bundle! In fact, it is the unique crepant resolution of $\mathbb{C}^2/(\mathbb{Z}/3)$ (the Kleinian singularity of type A_2), whose exceptional fiber is a union of two projective lines that touch at a point. If you know enough algebraic geometry to understand this assertion, then go ahead and try to prove it!

4. In the lecture, I asserted that

$$\operatorname{gr}\Big(U(\mathcal{V}) \otimes_{\operatorname{Sym}\mathfrak{k}} \mathbb{C}_\lambda\Big) \cong \Big(\operatorname{gr} U(\mathcal{V})\Big) \otimes_{\operatorname{Sym}\mathfrak{k}} \mathbb{C}_0$$

for any $\lambda \in \mathfrak{k}^*$. Convince yourself that this is true! Start with the example at the end of the talk, where $U(\mathcal{V})$ is the subalgebra of $\mathbb{C}[x_1, \partial_1, x_2, \partial_2]$ generated by $x_i\partial_j$, and $\mathfrak{k}$ is the one-dimensional subspace generated by $x_1\partial_1 + x_2\partial_2$. If you are not yet completely convinced, try the example in Problem 1 (note that η is irrelevant here).