WARTHOG 2017

Duality in Linear Programming

Review: A polarized arrangement is a triple

$\mathcal{V} = (V, \eta, \xi)$, where $V \subset \mathbb{R}^I$

$\eta \in \mathbb{R}^I / V \cong (V^\perp)^*$

$\xi \in V^* \cong (\mathbb{R}^I)^* / V^\perp$

$$0 \leftarrow \mathbb{R}^I/V \leftarrow \mathbb{R}^I \leftarrow V \leftarrow 0$$

$$0 \longrightarrow V^\perp \longrightarrow (\mathbb{R}^I)^* \longrightarrow V^* \longrightarrow 0$$

$$0 \longrightarrow V_{\mathbb{C}}^\perp \longrightarrow (\mathbb{C}^I)^* \longrightarrow V_{\mathbb{C}}^* \longrightarrow 0$$

$$\Big\uparrow \text{Lie}(-)$$

$$1 \longrightarrow K \longrightarrow (\mathbb{C}^\times)^I \longrightarrow T \longrightarrow 1$$

$K \subset (\mathbb{C}^\times)^I \curvearrowright D = \mathbb{C}\langle x_i, \partial_i \mid i \in I \rangle$

$U := D^K$

$Z(U) \cong \operatorname{Sym} \mathfrak{k} = \operatorname{Sym} V_{\mathbb{C}}^\perp$

$\eta : \operatorname{Sym} V_{\mathbb{C}}^\perp \longrightarrow \mathbb{C}$ $\qquad U_\eta := U \otimes_{Z(U)} \mathbb{C}$ $\qquad$ central quotient

We also have $\mu : \operatorname{Sym} \mathfrak{t} \longrightarrow U_\eta$.

$$\begin{array}{c}\operatorname{Sym} \mathfrak{t} \\ \parallel \\ \operatorname{Sym} V_{\mathbb{C}}^*\end{array}$$

$[\mu(\xi), -]$ defines a $\mathbb{Z}$-grading $U_\eta = \bigoplus_{k \in \mathbb{Z}} U_\eta^k$

Def: $\mathcal{O}(\mathcal{V})$ = f.g. U_η-modules on which $U_\eta^{\geq 0}$ acts locally finitely.

Let $V_\eta = V + \eta \subset \mathbb{R}^I$

$\forall \alpha \in \{\pm 1\}^I$, let $\Delta_\alpha := V_\eta \cap \{x \in \mathbb{R}^I \mid \alpha(i) x_i \geq 0 \ \forall i\}$

$\Sigma_\alpha := V \cap \{x \in \mathbb{R}^I \mid \alpha(i) x_i \geq 0 \ \forall i\}$.

Def: $\mathcal{F} = \{\alpha \mid \Delta_\alpha \neq \emptyset\}$ "feasible"

$\mathcal{B} = \{\alpha \mid \xi(\Sigma_\alpha)$ is bounded above$\}$ "bounded"

Prop: Simples in $\mathcal{O}(\mathcal{V})$ are indexed by $\mathcal{P} := \mathcal{F} \cap \mathcal{B}$

Def: $\mathcal{V}^! := (V^\perp, -\xi, -\eta)$ "Gale dual"

$V^\perp \subset (\mathbb{R}^I)^*$, $\quad -\xi \in (\mathbb{R}^I)^*/V^\perp \cong V^*$, $\quad -\eta \in (V^\perp)^* \cong \mathbb{R}^I/V$

The goal of this lecture is to prove the following result.

Thm: $F^! = B$ and $B^! = F$, so $P^! = P$.

Ex: $I = [3]$

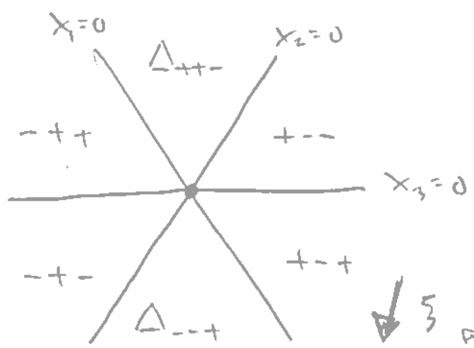
$V = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 0\}$

$\eta := [1, 0, 0] \in \mathbb{R}^3 / V$

$V^\perp = \mathbb{R}_\Delta \hookrightarrow (\mathbb{R}^3)^*$

$\xi := [0, 1, 2] \in (\mathbb{R}^3)^* / V^\perp$

V:

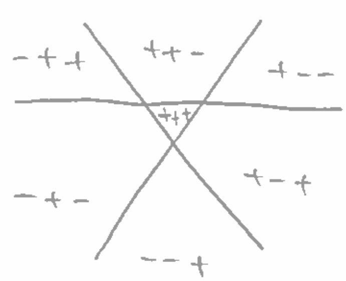


$\bullet = \Delta_{+++} = \Delta_{---}$

$B = \begin{array}{cc} ++- & +++ \\ +-- & --- \end{array}$

ξ really likes it when x_3 is big. After that, it looks at x_2.

V_η:



$\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) \in V_\eta$, so $\Delta_{+++} \neq \emptyset$

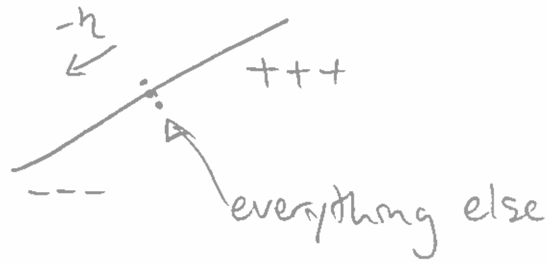
But $\Delta_{---} = \emptyset$.

F = those 7 sign vectors

$P = B \cap F = \begin{array}{cc} ++- & +++ \\ +-- & \end{array}$

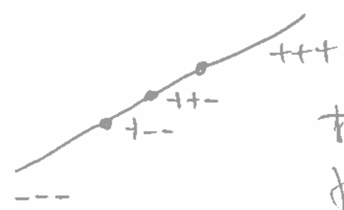
Now let's look at the dual

$V^{\perp}$:



$B^! =$ everything but $---$
$= F$

$V^{\perp}_{-\xi}$:



$-\xi : [0,-1,-2]$, so the first coord is always the biggest, then the second, then the third

$$F^! : \begin{matrix} +++ & ++- \\ --- & +-- \end{matrix} = B$$

Now let's prove the theorem. The proof will involve some very concrete and elementary linear algebra, following Chapters 1.3 & 1.4 of Ziegler's lectures on Polytopes.

Let $A \in \mathrm{Hom}(\mathbb{R}^d, \mathbb{R}^m)$, $z \in \mathbb{R}^m$.

Let $P(A,z) = \{x \in \mathbb{R}^d \mid Ax \leq z\}$

coordinate-wise

Intersection of m half-spaces
Polyhedron in general; cone if $z=0$.

For any k, have $\pi_k : \mathbb{R}^d \to \mathbb{R}^{d-1}$
"forget k^{th} coord"

For any $P \subset \mathbb{R}^d$, let $E_k P = \pi_k^{-1} \pi_k P$

$= \{x \in \mathbb{R}^d \mid \exists\, t \in \mathbb{R}$ st $x + te_k \in P\}$.

Lemma (Fourier-Motzkin Elimination): $\forall\, A \in \mathrm{Hom}(\mathbb{R}^d, \mathbb{R}^m)$, $\exists\, m' \in \mathbb{N}$ and $C_k \in \mathrm{Hom}(\mathbb{R}^m, \mathbb{R}^{m'})$ <u>with non-neg entries</u> st
$E_k P(A,0) = P(C_k A, 0)$.

More plainly: The linear inequalities defining $E_k P$ (cone) are non-neg linear combinations (m' of them) of the linear inequalities defining P

Pf: Let B be a matrix with the following rows:

- $\{a_i \mid a_{ik} = 0\}$ (the i^{th} row of A, if it kills e_k)

- $\{a_{ik} a_j + (-a_{jk}) a_i \mid a_{ik} > 0 \text{ and } a_{jk} < 0\}$

a pos. comb. of the i^{th} and j^{th} rows, which now kills e_k.

Then $B = C_k A$ for some C_k with non-neg entries.
I claim that $P(B,0) = E_k P(A,0)$.

Since the rows of B are non-neg combinations of the rows of A, we have $Bx \le 0 \ \forall x$ st $Ax \le 0$.
Thus $P(B,0) \supset P(A,0)$.

Since $Be_k = 0$, we have $P(B,0) = E_k P(B,0) \supset E_k P(A,0)$.

Now we need the opposite inclusion.
Suppose $x \in P(B,0)$; wts $\exists t \in \mathbb{R}$ st $x - te_k \in P(A,0)$,

If $a_{ik} = 0$, then a_i is also a row of B, so $a_i x \le 0 = a_{ik} t$.

ie $\quad a_i x - a_{ik} t \le 0 \quad \forall i$

ie $\quad a_i x \le a_{ik} t \quad \forall i$

ie $\quad \frac{1}{a_{ik}} a_i x \le t$ if $a_{ik} > 0$ and $\frac{1}{a_{jk}} a_j x \ge t$

Choose the worst i (maximizing $\frac{1}{a_{ik}} a_i x$ for $a_{ik} > 0$)

and the worst j (minimizing $\frac{1}{a_{jk}} a_j x$ for $a_{jk} < 0$)

Since $x \in P(B, 0)$, $(a_{ik} a_j + (-a_{jk}) a_i) x \leq 0$,

$$\text{so} \quad a_{ik} a_j x \leq a_{jk} a_i x$$

$$\text{so} \quad \frac{1}{a_{jk}} a_j x \geq \frac{1}{a_{ik}} a_i x$$

because $a_{ik} a_{jk} < 0$

So such a t exists! ✓

Lemma (Farkas Lemma, version I): Let $A \in \mathrm{Hom}(\mathbb{R}^d, \mathbb{R}^m)$, $z \in \mathbb{R}^m$ be given. Exactly one of the following two statements is true:

① $\exists\, x \in \mathbb{R}^d$ st $Ax \leq z$ (ie $P(A, z) \neq \emptyset$)

② $\exists\, c \in (\mathbb{R}^m)^*$ st $\underset{(\mathbb{R}^m)^*}{c \geq \underline{0}}$, $\underset{(\mathbb{R}^d)^*}{cA \geq \underline{0}}$, $\underset{\mathbb{R}}{cz < 0}$

Pf: First we show that ① and ② can't both hold. If they did, then $0 = \underline{0}x = (cA)x = c(Ax) \leq cz < 0 \Rightarrow\Leftarrow$

Now assume that ① fails, ie $P(A,z) = \emptyset$.

Let $Q = P((-z, A), 0) \subset \mathbb{R} \times \mathbb{R}^d$.

$\in \mathrm{Hom}(\mathbb{R}\times\mathbb{R}^d, \mathbb{R}^m)$

Claim: If $(t,x) \in Q$, then $t \le 0$.

Pf: Suppose $-zt + Ax \le 0$ and $t > 0$.

Then $Ax \le zt$

$\Rightarrow A(t^{-1}x) \le z$

$\Rightarrow t^{-1}x \in P, \Rightarrow \Leftarrow$. ✓

So $Q \subset \mathbb{R}_{\le 0} \times \mathbb{R}^d$.

Then $\underbrace{E_1 E_2 \cdots E_d}_{\text{everything but the first coord}} Q \subset \mathbb{R}_{\le 0} \times \mathbb{R}^d$. $(*)$

By FM elimination, $\exists\ C_1, \ldots, C_d$ with non-neg entries

st $E_1 \cdots E_d Q = P(C_1 \cdots C_d(-z, A), 0)$.

By $(*)$, one of the rows of $C_1 \cdots C_d(-z,A)$

must be $(t,x) \overset{\psi}{\longmapsto} \gamma x$ for some $\gamma > 0$.

Let $c \in (\mathbb{R}^m)^*$ be that row. Then $c(-z,A) = \psi$

$\Rightarrow cA = 0$ and $cz = -\gamma < 0$. ✓

Lemma (Farkas lemma, version II): Let $A \in \mathrm{Hom}(\mathbb{R}^d, \mathbb{R}^m)$, $z \in \mathbb{R}^m$ be given. Exactly one of the following is true:

① $\exists\, x \in \mathbb{R}^d$ with $Ax = z$ and $x \geq 0$.

② $\exists\, c \in (\mathbb{R}^m)^*$ with $\underset{\substack{\cap \\ (\mathbb{R}^d)^*}}{cA} \geq 0$ and $\underset{\substack{\cap \\ \mathbb{R}}}{cz} < 0$

Pf: ① $\iff \exists\, x$ st $Ax \leq z$, $(-Ax) \leq -z$, $-x \leq 0$

$$\iff \exists\, x \text{ st } \begin{pmatrix} A \\ -A \\ -I_d \end{pmatrix} x \leq \begin{pmatrix} z \\ -z \\ 0 \end{pmatrix}$$

$$\overset{\text{FL I}}{\iff} \nexists\, \underset{\substack{\cap \\ (\mathbb{R}^m)^*}}{c_1} \geq 0,\ \underset{\substack{\cap \\ (\mathbb{R}^m)^*}}{c_2} \geq 0,\ \underset{\substack{\cap \\ (\mathbb{R}^d)^*}}{b} \geq 0 \text{ st}$$

$$(c_1, c_2, b)\begin{pmatrix} A \\ -A \\ I_d \end{pmatrix} = 0 \text{ and } (c_1, c_2, b)\begin{pmatrix} z \\ -z \\ 0 \end{pmatrix} < 0$$

$$\iff \nexists\, c_1 \geq 0,\ c_2 \geq 0,\ b \geq 0 \text{ st}$$

$$(c_1 - c_2)A - b = 0, \quad (c_1 - c_2)z < 0$$

$$\iff \nexists\, c,\ b \geq 0 \text{ st}$$

$$cA - b = 0, \quad cz < 0$$

$$\iff \nexists\, c \text{ st } cA \geq 0,\ cz < 0. \qquad \checkmark$$

$$\iff \neg ②$$

Pf of Theorem:

Can reduce to the case where $\alpha = (+1)^I$;
wts that $\alpha \in F \iff \alpha \in B^!$.

Have

$$\begin{tikzcd} 0 \arrow[r] & V \arrow[r] & \mathbb{R}^I \arrow[r] \arrow[d, equal] & \mathbb{R}^I/V \arrow[r] \arrow[d, equal] & 0 \\ & & \mathbb{R}^d \arrow[r, "A"] & \mathbb{R}^m & \end{tikzcd}$$

and have

$$\begin{array}{ccc} \mathbb{R}^I/V & \cong & \mathbb{R}^m. \\ \Psi & & \Psi \\ h & =: & z \end{array}$$

FL II: Either ① $\exists\ x \in \mathbb{R}^I_{\geq 0}$ lifting h
or ② $\exists\ c \in \underbrace{(\mathbb{R}^I/V)^*}_{= V^\perp}$ with $cA \geq 0$ and $cz < 0$.

① $\iff \exists\ x \in \Delta_\alpha \iff \alpha \in F$

② $\iff \exists\ c \in \Sigma^!_\alpha$ with $\langle c, h \rangle < 0 \iff -h$ unbounded on $\Sigma^!_\alpha$

$\iff \alpha \notin B^!$.

Done!