WARTHOG 2017

The hypertoric enveloping algebra I: Higgs description

This whole workshop is about the representation theory of the hypertoric enveloping algebra. My goal in this talk is to define the hypertoric enveloping algebra and get you comfortable working with it. No modules will be discussed.

The hypertoric enveloping algebra is defined as a subalgebra of the Weyl algebra, so we'll start with that.

Fix n.

Let $D = \mathbb{C}\langle x_1, \partial_1, \ldots, x_n, \partial_n \rangle \Big/ \Big\langle \begin{matrix} [x_i, x_j] \\ [\partial_i, \partial_j] \\ [\partial_i, x_j] - \delta_{ij} \end{matrix} \Big\rangle$

$= \{$polynomial differential operators on $\mathbb{C}[x_1, \ldots, x_n]\}$

Filtration: $F_k D := \left\{ \begin{matrix} \text{stuff that can be written as a} \\ \text{poly of degree} \leq k \text{ in } x_1, \partial_1, \ldots, x_n, \partial_n \end{matrix} \right\}$

eg $F_0 D = \mathbb{C}$
$\quad \cap$
$F_1 D = \mathbb{C}\{1, x_1, \partial_1, \ldots, x_n, \partial_n\}$
$\quad \cap$
$\quad \vdots$

$$\operatorname{gr} D := \bigoplus_{k \geq 0} F_k D / F_{k-1} D \cong \mathbb{C}[x_1, y_1, \dots, x_n, y_n]$$

Grading: $(\mathbb{C}^\times)^n \circlearrowright D$

$$t = (t_1, \dots, t_n) \in (\mathbb{C}^\times)^n \rightsquigarrow \begin{aligned} t \cdot x_i &= t_i x_i \\ t \cdot \partial_i &= t_i^{-1} \partial_i . \end{aligned}$$

(so that our relation is homogenious)

$\forall z \in \mathbb{Z}^n$, let $D_z := \{ a \in D \mid t \cdot a = t_1^{z_1} \cdots t_n^{z_n} a \ \ \forall t \in (\mathbb{C}^\times)^n \}$

z-eigenspace

eg $D_0 = ?$ $\quad h_i^+ := x_i \partial_i \in D_0$

$\quad\quad\quad\quad h_i^- := \partial_i x_i = h_i^+ + 1 \in D_0$

$$D_0 \supset \mathbb{C}[h_1^+, \dots, h_n^+] = \mathbb{C}[h_1^-, \dots, h_n^-].$$

Is this all of D_0? How about $x_1^2 \partial_1^3 x_1 x_2^4 \partial_2^4$?

Can we write this in terms of $h_1^\pm, h_2^\pm$?

We can, and we'll prove it with a couple of lemmas.

Lemma 1: If $a \in D_z$, then $[h_i^\pm, a] = z_i a$.

Pf: If $a = x_j$ for $j \neq i$, $[h_i^\pm, a] = 0 = z_i a$ $(z_i = 0)$

If $a = x_i$, $[h_i^\pm, a] = [x_i \partial_i, x_i]$

$$= x_i \partial_i x_i - x_i^2 \partial_i$$
$$= x_i (x_i \partial_i + 1) - x_i^2 \partial_i$$
$$= x_i$$
$$= z_i a \quad (z_i = 1)$$

Similarly $[h_i^\pm, \partial_j] = -\delta_{ij} \partial_j = z_i \partial_j$.

Now I want to say that we're done because I've checked all of the generators.

If the lemma holds for $a \in D_z$ and $a' \in D_{z'}$, then it also holds for $aa' \in D_{z+z'}$:

$$[h_i^\pm, aa'] = [h_i^\pm, a]a' + a[h_i^\pm, a']$$
$$= z_i aa' + a z_i' a'$$
$$= (z_i + z_i') aa'. \quad \checkmark$$

Lemma 2: $\forall\, k \geq 0$, $\quad \partial_i^k x_i^k = (h_i^+ + 1)\cdots(h_i^+ + k)$

$$x_i^k \partial_i^k = (h_i^- - 1)\cdots(h_i^- - k)$$

Pf of 1st statement: Induct on k $\quad k=0$ is clear

$$\begin{aligned}
\partial_i^{k+1} x_i^{k+1} &= \partial_i^k (\partial_i x_i)\, x_i^k \\
&= \partial_i^k h_i^- x_i^k \\
&= \partial_i^k [h_i^-, x_i^k] + \partial_i^k x_i^k h_i^- \\
&= k\, \partial_i^k x_i^k + \partial_i^k x_i^k h_i^- \\
&= \partial_i^k x_i^k (h_i^- + k) \\
&= \partial_i^k x_i^k (h_i^+ + (k+1)).
\end{aligned}$$

Now use inductive hyp. ✓

Ex: $x_1^2 \partial_1^3 x_1 x_2^4 \partial_2^4 = (x_1^2 \partial_1^2)(\partial_1 x_1)(x_2^4 \partial_2^4)$

$$= (h_1^- - 1)(h_1^- - 2)(h_1^+ + 1)(h_2^- - 1)\cdots(h_2^- - 4)$$

Cor: $D_0 = \mathbb{C}[h_1^\pm, \_\_, h_n^\pm]$.

This is isomorphic to a polynomial ring, but it is naturally filtered rather than graded.

$\operatorname{gr} D_0 = \mathbb{C}[h_1, \_\_, h_n]$, where $h_i = x_i y_i$

How about D_z for $z \neq 0$?

$\forall\, z \in \mathbb{Z}^n$, let $(z_+)_i = \max(z_i, 0)$

$$(z_-)_i = \max(-z_i, 0),$$

so $z = z_+ - z_-$.

Ex: $z = (5, -2, 0, 1, 1)$

$$z_+ = (5, 0, 0, 1, 1)$$

$$z_- = (0, 2, 0, 0, 0).$$

Let $m^z := x^{z_+} \partial^{z_-} \in D_z$

Ex: $m^z = x_1^5 x_4 x_5 \partial_2^2 \in D_{(5,-2,0,1,1)}$

For any $f(h) = f(h_1^+, \dots, h_n^+) \in D_0$
and any $z \in \mathbb{Z}^n$, let

$$f_z(h) = f(h_1^+ + z_1, \dots, h_n^+ + z_n).$$

Lemma 3: $f(h)\, m^z = m^z f_z(h)$
$$m^z f(z) = f_{-z}(h)\, m^z$$

Pf: Reduce to the case where $f(h) = h_i^+$ and $m^z = x_j$ or ∂_j

$$\begin{aligned} h_i^+ x_j &= [h_i^+, x_j] + x_j h_i^+ \\ &= \delta_{ij} x_j + x_j h_i^+ \\ &= x_j (h_i^+ + \delta_{ij}) \end{aligned}$$

So switching the order of h_i^+ and x_j is free if $i \neq j$ and adds 1 to h_i^+ if $i = j$. ∂_j is similar. ✓

Cor: $\forall z \in \mathbb{Z}^n$, $D_z = \mathbb{C}[h_1^{\pm}, \ldots, h_n^{\pm}]\, m^z$

Pf: Use lemmas 2 and 3 to move everything to the left.

Ex: $n=1$, $x_1^8 \partial_1^5 \in D_3$

$$\begin{aligned} x_1^8 \partial_1^5 &= x_1^3 (x_1^5 \partial_1^5) \\ &= x_1^3 (h_1^- - 1)(h_1^- - 2) \cdots (h_1^- - 5) \\ &= (h_1^- - 4) \cdots (h_1^- - 8)\, x_1^3 \end{aligned}$$

Now we have $D = \bigoplus_{z \in \mathbb{Z}^n} D_z = \bigoplus_{z \in \mathbb{Z}^n} \mathbb{C}[h_1^\pm, \dots, h_n^\pm] \cdot m^z$.

How do we multiply?

$$f(h) m^z \cdot g(h) m^w = f(h) \left(m^z g(h) \right) m^w$$
$$= f(h) g_{-z}(h) m^z \cdot m^w$$
$$= ?$$

It's not true that $m^z \cdot m^w = m^{z+w}$.

Ex: $n=1$: $m^1 = x_1$, $m^{-1} = \partial_1$

$m^1 \cdot m^{-1} = x_1 \partial_1 = h_1^+ \neq 1 = m^0$.

Prop (explained to me by Justin Hilburn):

Let $[h_i]^k = \begin{cases} 1 & \text{if } k=0 \\ x_i^k \partial_i^k = (h_i^- - 1) \cdots (h_i^- - k) & \text{if } k>0 \\ \partial_i^{-k} x_i^{-k} = (h_i^+ + 1) \cdots (h_i^+ - k) & \text{if } k<0 \end{cases}$

Then $m^z \cdot m^w = \left(\prod_{\substack{z_i w_i < 0 \\ |z_i| \le |w_i|}} [h_i]^{z_i} \right) m^{z+w} \left(\prod_{\substack{z_i w_i < 0 \\ |z_i| > |w_i|}} [h_i]^{-w_i} \right)$.

Pf: Exercise.

Now we understand the Weyl algebra pretty well, and we're ready to define the hypertoric enveloping algebra.

Input: Subtorus $K \subset (\mathbb{C}^*)^n$, not containing any coordinate subtorus.

Let $T := (\mathbb{C}^*)^n / K$

$$1 \to K \to (\mathbb{C}^*)^n \to T \to 1$$

$$0 \to \mathfrak{k} \to \mathbb{C}^n \to \mathfrak{t} \to 0$$

$$0 \leftarrow \mathfrak{k}^* \leftarrow (\mathbb{C}^n)^* \leftarrow \mathfrak{t}^* \leftarrow 0$$

$$\begin{array}{ccccccccc} 0 \leftarrow & \mathfrak{k}^*_{\mathbb{Z}} & \leftarrow & \mathbb{Z}^n & \leftarrow & \mathfrak{t}^*_{\mathbb{Z}} & \leftarrow 0 \\ & \| & & \| & & \| & \\ & \mathrm{Hom}(K, \mathbb{C}^*) & & \mathrm{Hom}((\mathbb{C}^*)^n, \mathbb{C}^*) & & \mathrm{Hom}(T, \mathbb{C}^*) & \end{array}$$

Def: $U := D^K = \displaystyle\bigoplus_{z \in \mathfrak{t}^*_{\mathbb{Z}} \subset \mathbb{Z}^n} \mathbb{C}[h_1^{\pm}, \dots, h_n^{\pm}]\, m^z$.

Let's think about a concrete example.

Ex: $n=3$, $K = \mathbb{C}^\times_\Delta \hookrightarrow (\mathbb{C}^\times)^3$

$$T = (\mathbb{C}^\times)^3 / \mathbb{C}^\times_\Delta \cong (\mathbb{C}^\times)^2$$

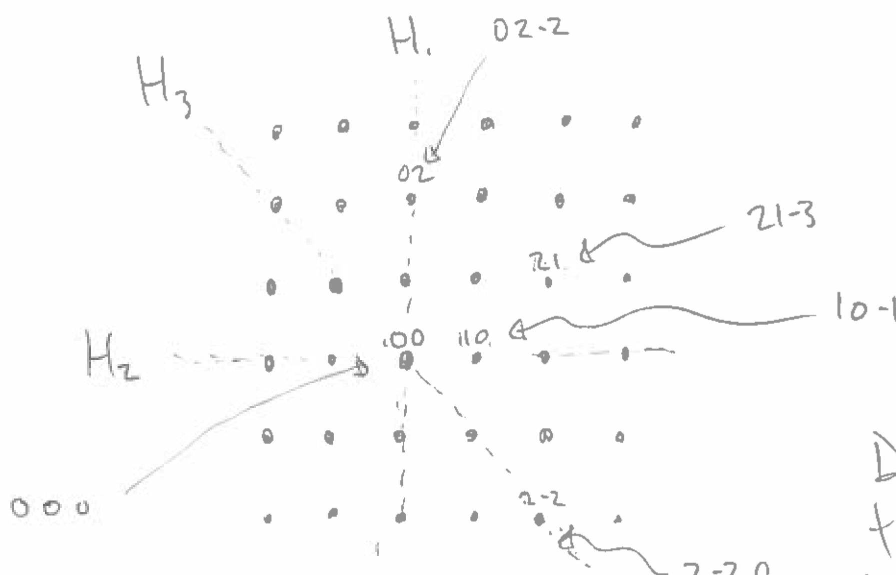
$$1 \to \mathbb{C}^\times \xrightarrow{\Delta} (\mathbb{C}^\times)^3 \to (\mathbb{C}^\times)^2 \to 1$$

$$t \mapsto (t,t,t)$$

$$(t_1,t_2,t_3) \mapsto (t_1/t_3,\ t_2/t_3)$$

$$0 \to \mathbb{C} \xrightarrow{\begin{bmatrix}1\\1\\1\end{bmatrix}} \mathbb{C}^3 \xrightarrow{\begin{bmatrix}1 & 0 & -1\\ 0 & 1 & -1\end{bmatrix}} \underset{\shortparallel}{\mathbb{C}^2}_{\mathfrak{t}} \to 0$$

$$\begin{array}{ccccccccc} 0 & \leftarrow & (\mathbb{C})^* & \xleftarrow{[1\ 1\ 1]} & (\mathbb{C}^3)^* & \xleftarrow{\begin{bmatrix}1 & 0\\ 0 & 1\\ -1 & -1\end{bmatrix}} & (\mathbb{C}^2)^* & \leftarrow & 0 \\ & & \cup & & \cup & & \cup & & \\ 0 & \leftarrow & \mathbb{Z} & \leftarrow & \mathbb{Z}^3 & \leftarrow & \mathbb{Z}^2 & \leftarrow & 0 \end{array}$$



$$\begin{array}{ccc} \mathbb{Z}^2 & \hookrightarrow & \mathbb{Z}^3 \\ \shortparallel & & \shortparallel \\ \mathfrak{t}^*_{\mathbb{Z}} & & \end{array}$$

Dotted lines are the coordinate hyperplanes in $(\mathbb{C}^3)^*$

$H_i := \mathfrak{t}^* \cap \{i^{th}$ coord hyperplane in $(\mathbb{C}^3)^*\}$

Basis element for each lattice point

$$m^{21} \cdot m^{2-2} = \;?$$

$$\begin{array}{cc} \| & \| \\ m^{21-3} & m^{2-20} \\ \| & \| \\ x_1^2 x_2 \partial_3^3 & x_1^2 \partial_2^2 \end{array}$$

Observe that these are K-invariant!

$$m^{21-3} \cdot m^{2-20} = [h_2]^1 \, m^{4-1-3}$$

$$= (h_2^- - 1)\, m^{4-1-3}$$

$$= (h_2^- - 1)\, m^{4-1}$$

Funny business happened only in 2^{nd} coordinate because $(2,1)$ and $(2,-2)$ lie on opposite sides of H_2, but not H_1 or H_3.

Summary: $\mathcal{U} = \bigoplus_{z \in \mathfrak{t}^*_{\mathbb{Z}}} \mathbb{C}[h_1^\pm, \ldots, h_n^\pm] \cdot m^z$.

$$m^z \cdot m^w = (\sim)\, m^{z+w} (\sim),$$

where correction factors involve those $h_i^\pm$ for which z and w lie on opposite sides of H_i

Last question: What is the center of $\mathcal{U}$?

- Recall: If $a \in D_z$, then $[h_i^+, a] = z_i a$.

  Thus if $a \neq 0$ and $z_i \neq 0$, $[h_i^+, a] \neq 0$, so a is not central. Thus $Z(\mathcal{U}) \subset D_0$.

- Recall: $f(h) m^z = m^z f_z(h)$.

  Thus $f(h) \in D_0$ is central iff

  $f(h) = f_z(h) \quad \forall z \in \mathfrak{t}_{\mathbb{Z}}^* \subset \mathbb{Z}^n$

Ex: In our example, $Z(\mathcal{U}) = \mathbb{C}[h_1^+ + h_2^+ + h_3^+]$

In general, $Z(\mathcal{U}) \cong \operatorname{Sym} \mathfrak{k}$

$\subset \operatorname{Sym} \mathbb{C}^n$

$= \mathbb{C}[h_1^+, \dots, h_n^+] = D_0$

Ex: If $K = (\mathbb{C}^\times)^n$, $\mathcal{U} = D^K = D_0$ is abelian

If $K = \{1\}$, $\mathcal{U} = D$ and $Z(\mathcal{U}) = \mathbb{C}$

The bigger K is, the smaller $\mathcal{U}$ is

and the bigger $Z(\mathcal{U})$ is.